A density tensor hierarchy for open system dynamics: retrieving the noise

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Abstract
We develop a density tensor hierarchy for open system dynamics that recovers information about fluctuations (or 'noise') lost in passing to the reduced density matrix. For the case of fluctuations arising from a classical probability distribution, the hierarchy is formed from expectations of products of pure state density matrix elements and can be compactly summarized by a simple generating function. For the case of quantum fluctuations arising when a quantum system interacts with a quantum environment in an overall pure state, the corresponding hierarchy is defined as the environmental trace of products of system matrix elements of the full density matrix. Whereas all members of the classical noise hierarchy are system observables, only the lowest member of the quantum noise hierarchy is directly experimentally measurable. The unit trace and idempotence properties of the pure state density matrix imply descent relations for the tensor hierarchies, that relate the order n tensor, under contraction of appropriate pairs of tensor indices, to the order $n - 1$ tensor. As examples to illustrate the classical probability distribution formalism, we consider a spatially isotropic ensemble of spin-$1/2$ pure states, a quantum system evolving by an Itô stochastic Schrödinger equation and a quantum system evolving by a jump process Schrödinger equation. As examples to illustrate the corresponding trace formalism in the quantum fluctuation case, we consider the tensor hierarchies for collisional Brownian motion of an infinite mass Brownian particle and for the weak coupling Born–Markov master equation. In different specializations, the latter gives the hierarchies generalizing the quantum optical master equation and the Caldeira–Leggett master equation. As a further application of the density tensor, we contrast stochastic Schrödinger equations that reduce and that do not reduce the state vector, and discuss why a quantum system coupled to a quantum environment behaves like the latter. The descent relations for our various examples are checked in a series of appendices.

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1. Introduction

Increasing attention is being paid to the dynamics of open quantum systems, that is, to quantum systems acted on by an environment. Such systems are of interest for studies of dissipative phenomena, decoherence, backgrounds to quantum computers and to precision measurements, and theories of quantum measurement. A principal tool in studying open quantum systems is the reduced density matrix, obtained from the pure state density matrix by tracing over environment degrees of freedom, or in stochastic models where the environment is represented by a noise term in the Schrödinger equation, by averaging over the noise. As is well known, this transition from the pure state density matrix to the reduced density matrix is not one-to-one, since information about the total system is lost. For example, in stochastic models, there is known to be a continuum of different unravellings, or pure state density matrix stochastic evolutions, that yield the same master equation for the reduced density matrix. The question that we investigate here is the extent to which one can form objects that refer only to the basis vectors of the system Hilbert space, but that nonetheless recapture information that is lost in passing to the reduced density matrix. In the first part of this paper (sections 2–5), we discuss classical noise arising from fluctuations defined by classical probability distributions. In the second part (sections 6–9), we give an analogous discussion of quantum noise, which appears in the physically important case of a quantum system coupled to a quantum environment in an overall pure state. We also give an extension, making contact with the discussion of the first part, to the case in which the overall system is in a mixed state superposition of pure states. The final section contains a discussion of quantum measurements that relates the material in the first and second parts.

For the case of classical probability distributions, a relevant discussion appears in chapter 5 of the book *The Theory of Open Quantum Systems* by Breuer and Petruccione [1], following up on earlier papers by those authors [2], by Wiseman [3] and by Mølmer, Castin and Dalibard [4]. In simplified form, Breuer and Petruccione introduce an ensemble of pure state vectors $|\psi_\alpha\rangle$, each drawn from the same system Hilbert space $\mathcal{H}_S$, with each vector assumed to occur in the ensemble with probability w_α, $\sum_\alpha w_\alpha = 1$. Measurement of a general self-adjoint operator R for a system prepared in $|\psi_\alpha\rangle$ typically gives a range of values, the mean of which is given by $\langle\psi_\alpha|R|\psi_\alpha\rangle$. The mean or expectation over the ensemble of pure state vectors is then given by

$$\sum_\alpha w_\alpha \langle\psi_\alpha|R|\psi_\alpha\rangle = \operatorname{Tr}\rho R, \tag{1a}$$

with ρ the mixed state or reduced density matrix defined by

$$\rho = \sum_\alpha w_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|. \tag{1b}$$

Breuer and Petruccione point out that there are three variances that are relevant. The variance of measurements of R over all pure states in the ensemble is given by

$$\operatorname{Var}(R) = \operatorname{Tr}\rho(R - \operatorname{Tr}\rho R)^2 = \operatorname{Tr}\rho R^2 - (\operatorname{Tr}\rho R)^2. \tag{2a}$$

This can be written as the sum of two non-negative terms,

$$\operatorname{Var}(R) = \operatorname{Var}_1(R) + \operatorname{Var}_2(R), \tag{2b}$$

with $\operatorname{Var}_1(R)$ the ensemble average of the variances of R within each pure state of the ensemble,

$$\operatorname{Var}_1(R) = \sum_\alpha w_\alpha[\langle\psi_\alpha|R^2|\psi_\alpha\rangle - \langle\psi_\alpha|R|\psi_\alpha\rangle^2], \tag{2c}$$

and with $\mathrm{Var}_2(R)$ the variance of the pure state means of R over the ensemble,

$$\mathrm{Var}_2(R) = \sum_\alpha w_\alpha \langle \psi_\alpha | R | \psi_\alpha \rangle^2 - \left[\sum_\alpha w_\alpha \langle \psi_\alpha | R | \psi_\alpha \rangle \right]^2. \tag{2d}$$

Thus, $\mathrm{Var}_1(R)$ is an ensemble average of the quantum variances of R, while $\mathrm{Var}_2(R)$ is a measure of the spread of the average values of R resulting from the statistical properties of the ensemble. As Breuer and Petruccione note, neither of the subsidiary variances $\mathrm{Var}_{1,2}$ can be expressed as the density matrix expectation of some self-adjoint operator.

Our aim in the first part of this paper is to extend the formalism of [1] by utilizing a density tensor hierarchy, which captures the statistical information that is lost in forming the reduced density matrix of equation (1b). A density tensor, defined as an ensemble average of density matrices, was first introduced by Mielnik [5], and was applied to discussions of density functions on the space of quantum states and their application to thermalization of quantum systems by Brody and Hughston [6]. These papers, in addition to introducing the concept of a density tensor which is developed further here, also contain the important result that in the case of a continuum probability distribution the density tensor hierarchy gives all of the information needed to reconstruct the probability function w_α. In particular, the variances $\mathrm{Var}_{1,2}$ for any observable, and more general statistical properties of the ensemble as well, can be expressed as contractions of density tensor matrix elements with appropriate matrix elements of the observable(s) of interest.

The basic construction of the density tensor hierarchy corresponding to a classical probability distribution $\{w_\alpha\}$ is given in section 2. Here, we generalize the reduced density matrix of equation (1b) to a density tensor, formed by taking a product of pure state density matrix elements and averaging over the ensemble of pure states. When the ψ_α are independent of α, this tensor reduces to an n-fold product of reduced density matrices, and so the difference between the density tensor and this product is a measure of the statistical fluctuations in the ensemble. In the generic case of non-trivial dependence of ψ_α on α, there are some general statements that can be made. First of all, the order n density tensor is a symmetric tensor in its pair indices, and it can be considered as a matrix operator acting on the n-fold tensor product of the system Hilbert space $\mathcal{H}_S$ with itself. The symmetry of the density tensor allows construction of a generating function that on expansion gives the density tensors of all orders. Additionally, as a consequence of the unit trace and idempotence conditions obeyed by the pure state density matrix, the density tensor hierarchy satisfies a system of descent equations, relating the order n tensor to the order $n-1$ tensor when any row index is contracted with any column index. We show that the variances $\mathrm{Var}_{1,2}$ defined by Breuer and Petruccione can be expressed in terms of appropriate contractions of density tensor elements with operator matrix elements.

In subsequent sections, we develop some concrete applications of the general formalism for classical probability distributions. In section 3, we consider an isotropic ensemble of spin-$1/2$ pure state density matrices, construct the density tensors through order 3, verify the descent equations and calculate the generating function. In section 4, we apply the formalism to a quantum system evolving under the influence of noise as described by a stochastic Schrödinger equation, with the ensemble defined as the set of all histories of an initial quantum state under the influence of the noise. Assuming white noise described by the Itô calculus, we give the dynamics of the general density tensor in terms of the general unravelling of the Lindblad equation constructed by Wiseman and Diósi [7] and show that the order two and higher density tensors distinguish between inequivalent unravellings that give the same reduced density matrix (i.e., the same order one density tensor). In section 5,

we develop an analogous formalism for the case of jump (piecewise deterministic process) unravellings of the Lindblad equation.

We turn next to an analysis of a quantum system coupled to a quantum environment, rather than to an external classical noise source. Here, one is confronted with the problem of discussing the system dissipation associated with the system–environment interaction within a single overall pure state of system plus environment (or in a thermal state that is a weighted average of such pure states). Typically, in master equation derivations, the system–environment interaction[1] H has vanishing expectation in the environment, but its square H^2 does not have a vanishing expectation, because the environment is not in an eigenstate of H. The associated variance is then a measure of quantum fluctuations associated with the environment state and is the source of quantum 'noise' driving the system dissipation. Our aim in the second part of this paper is to generalize the formalism of the first part to recapture information about this noise that is lost in the passage to the system reduced density matrix. We do this in section 6 by defining a density tensor hierarchy as the trace over the environment of a product of environment operators constructed as the system matrix elements of the total density matrix. Unlike the classical noise construction, which uses only the system density matrix, the construction in the quantum noise case requires knowledge of the full system plus environment density matrix, and so (except for the order one case) does not give a system observable. It is nonetheless computable in any theory of the system plus environment and is of theoretical, rather than empirical, interest. Because the environment operators entering the construction are non-commutative, this hierarchy is no longer totally symmetric in its system index pairs, but by the cyclic permutation property of the trace it is symmetric under cyclic permutation of the system index pairs. Also, because the system trace of these environment operators gives only the reduced environment density matrix, rather than unity, there is in general no descent equation associated with taking this trace. However, when indices of adjacent system operators are contracted, one gets the square of the overall density matrix, and so there remains a set of descent relations connecting the order (n) tensor to the order $(n-1)$ tensor. Finally, in the case of thermal (or other mixed) overall states, we define the appropriate tensor as a weighted sum of pure state tensors, in analogy with the definition of section 2.

In subsequent sections, we give applications of the trace hierarchy formalism to several classic problems discussed in the theory of quantum master equations. In section 7, we consider the quantum Brownian motion (and resulting decoherence) of a massive Brownian particle in interaction with an independent particle bath of scatterers. In section 8, we discuss the tensor hierarchy corresponding to the weak coupling Born–Markov master equation and its specialization to the quantum optical master equation. Finally in section 9, we give an analogous discussion for the Caldeira–Leggett model of a particle in interaction with a system of environmental oscillators.

We conclude with a discussion that bridges the considerations of the classical noise and quantum noise cases. In section 10, we contrast two different Itô stochastic Schrödinger equations, both of which have the same Lindblad, but only one of which leads to state vector reduction. We relate this to the fact that the equation giving the time derivative of the stochastic expectation of operator variances involves the order two density tensor, which differs for the two cases. We discuss the analogous equation for the time dependence of the variance of a 'pointer operator' in the case of a quantum system coupled to a quantum environment and show why this does not lead to state vector reduction. Thus, we see no mechanism for quantum

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[1] What we call H is usually denoted by H_I in the open systems literature. To avoid confusion, all other Hamiltonians will carry subscripts, e.g., H_S and $H_{\mathcal{E}}$ for the system and environment Hamiltonians, H_{TOT} for the total Hamiltonian, etc.

'noise' in a closed quantum system plus environment to provide a resolution of the quantum measurement problem.

2. The density tensor for classical noise and its kinematical properties

We proceed to establish our notation and to define the density tensor hierarchy in the classical noise case. We denote the pure state density matrix formed from the unit normalized state $|\psi_\alpha\rangle$ by ρ_α, with

$$\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|, \tag{3a}$$

and its general matrix element between states $|i\rangle$ and $|j\rangle$ of $\mathcal{H}_S$ by

$$\rho_{\alpha;ij} \equiv \langle i|\rho_\alpha|j\rangle. \tag{3b}$$

The unit trace condition on ρ_α states that

$$\mathrm{Tr}\,\rho_\alpha = \langle\psi_\alpha|\psi_\alpha\rangle = 1, \tag{3c}$$

and the idempotence condition on ρ_α states that

$$\rho_\alpha^2 = |\psi_\alpha\rangle\langle\psi_\alpha||\psi_\alpha\rangle\langle\psi_\alpha| = |\psi_\alpha\rangle\langle\psi_\alpha| = \rho_\alpha. \tag{3d}$$

We now define the order n density tensor by

$$\rho_{i_1 j_1, i_2 j_2, \ldots, i_n j_n}^{(n)} = \sum_\alpha w_\alpha \rho_{\alpha;i_1 j_1} \rho_{\alpha;i_2 j_2} \cdots \rho_{\alpha;i_n j_n} = E[\rho_{\alpha;i_1 j_1} \rho_{\alpha;i_2 j_2} \cdots \rho_{\alpha;i_n j_n}], \tag{4a}$$

with $E[F_\alpha]$ a shorthand for

$$E[F_\alpha] = \sum_\alpha w_\alpha F_\alpha. \tag{4b}$$

Since

$$\rho_{ij}^{(1)} = \sum_\alpha w_\alpha \rho_{\alpha;ij} = \sum_\alpha w_\alpha \langle i|\rho_\alpha|j\rangle, \tag{5a}$$

we see that this is just the $|i\rangle$ to $|j\rangle$ matrix element of the reduced density matrix ρ defined in equation (1b),

$$\rho_{ij}^{(1)} = \langle i|\rho|j\rangle, \tag{5b}$$

and so the density tensor of equation (4a) is a natural generalization of the usual reduced density matrix. When the states $|\psi_\alpha\rangle$ are independent of the label α, the definition of equation (4a) simplifies to

$$\rho_{i_1 j_1, i_2 j_2, \ldots, i_n j_n}^{(n)} = \rho_{i_1 j_1} \rho_{i_2 j_2} \cdots \rho_{i_n j_n}, \tag{5c}$$

and so the difference between equation (4a) and a product of reduced density matrix elements is a reflection of the statistical structure of the ensemble. Since the factors within the expectation $E[\cdots]$ on the right of equation (4a) are just ordinary complex numbers, the density tensor is symmetric under interchange of any index pair $i_l j_l$ with any other index pair $i_m j_m$. Consequently, we can define a generating function for the density tensor by

$$G[a_{ij}] = E[e^{\rho_{\alpha;ij} a_{ij}}] = \sum_{n=0}^\infty \frac{a_{i_1 j_1} \cdots a_{i_n j_n}}{n!} \rho_{i_1 j_1, \ldots, i_n j_n}^{(n)}, \tag{5d}$$

where repeated indices i, j are summed. It will often be convenient to abbreviate $\rho_{\alpha;ij} a_{ij}$ by $\rho_\alpha \cdot a$, so that the generating function becomes in this notation $G[a] = E[e^{\rho_\alpha \cdot a}]$.

Although the density tensor for $n > 1$ is not an operator on $\mathcal{H}_\mathcal{S}$, it clearly has the structure of an operator on the n-fold tensor product $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{S} \otimes \cdots \otimes \mathcal{H}_\mathcal{S}$. Motivated by this, we will often find it convenient to write the definition of equation (4a) as

$$\rho^{(n)} = E\left[\prod_{\ell=1}^{n} \rho_{\alpha;\ell}\right], \tag{5e}$$

with each factor $\rho_{\alpha;\ell}$ acting on a distinct factor Hilbert space $\mathcal{H}_{\mathcal{S};\ell}$ in the tensor product $\prod_{\ell=1}^{n} \mathcal{H}_{\mathcal{S};\ell}$. One can pass easily back and forth from this notation to one in which the system matrix indices are displayed explicitly.

Let us consider next the result of contracting any row index i_l with any column index j_k. There are two basic cases: (i) one can contract a row index i_l with its corresponding column index j_l and (ii) one can contract a row index i_l with a column index j_k with $k \neq l$. Since the density tensor is symmetric in its index pairs, it suffices to consider only one example of each case, since all others can be obtained by permutation. For the contraction of i_1 with j_1 we find

$$\delta_{i_1 j_1} \rho^{(n)}_{i_1 j_1, i_2 j_2, \ldots, i_n j_n} = E\left[(\mathrm{Tr}\,\rho) \rho_{\alpha; i_2 j_2} \cdots \rho_{\alpha; i_n j_n}\right] = E\left[\rho_{\alpha; i_2 j_2} \cdots \rho_{\alpha; i_n j_n}\right] = \rho^{(n-1)}_{i_2 j_2, \ldots, i_n j_n}, \tag{6a}$$

where we have used the unit trace condition of equation (3c). For the contraction of j_1 with i_2, we find

$$\delta_{j_1 i_2} \rho^{(n)}_{i_1 j_1, i_2 j_2, \ldots, i_n j_n} = E\left[(\rho^2)_{\alpha; i_1 j_2} \cdots \rho_{\alpha; i_n j_n}\right] = E\left[\rho_{\alpha; i_1 j_2} \rho_{\alpha; i_3 j_3} \cdots \rho_{\alpha; i_n j_n}\right] = \rho^{(n-1)}_{i_1 j_2, i_3 j_3, \ldots, i_n j_n}, \tag{6b}$$

where now we have used the idempotence condition of equation (3d). As an illustration of how this works when all possible index pair contractions are considered, we give the complete set of contractions reducing the second-order density tensor to a first-order density tensor,

$$\begin{aligned}
\delta_{i_1 j_1} \rho^{(2)}_{i_1 j_1, i_2 j_2} &= \rho^{(1)}_{i_2 j_2}, \\
\delta_{i_2 j_2} \rho^{(2)}_{i_1 j_1, i_2 j_2} &= \rho^{(1)}_{i_1 j_1}, \\
\delta_{j_1 i_2} \rho^{(2)}_{i_1 j_1, i_2 j_2} &= \rho^{(1)}_{i_1 j_2}, \\
\delta_{j_2 i_1} \rho^{(2)}_{i_1 j_1, i_2 j_2} &= \rho^{(1)}_{i_2 j_1}.
\end{aligned} \tag{7a}$$

Referring to the generating function of equation (5d), the general descent equations can be summarized compactly by the two identities,

$$\begin{aligned}
\delta_{mr} \frac{\partial G[a_{ij}]}{\partial a_{mr}} &= E[(\mathrm{Tr}\,\rho_\alpha)\, \mathrm{e}^{\rho_{\alpha;ij} a_{ij}}] = G[a_{ij}], \\
\delta_{rp} \frac{\partial^2 G[a_{ij}]}{\partial a_{mr} \partial a_{pq}} &= E[\rho_{mr} \rho_{rq}\, \mathrm{e}^{\rho_{\alpha;ij} a_{ij}}] = E[\rho_{mq}\, \mathrm{e}^{\rho_{\alpha;ij} a_{ij}}] = \frac{\partial G[a_{ij}]}{\partial a_{mq}}.
\end{aligned} \tag{7b}$$

When the density matrix ρ used to define the density tensor is a mixed state density matrix, the trace descent relation of equation (6a) is unchanged, while the indempotency relation of equation (6b) relates the contraction an order (n) tensor to an order $(n-1)$ tensor in which one factor ρ is replaced by ρ^2; this is not a member of the original hierarchy, but still gives a useful relation for checking calculations.

To conclude this section, let us return to the variances introduced by Breuer and Petruccione. In terms of the order one and order two density tensors, we evidently have

$$\begin{aligned}
\mathrm{Var}_1(R) &= \rho^{(1)}_{i_1 j_1} (R^2)_{j_1 i_1} - \rho^{(2)}_{i_1 j_1, i_2 j_2} R_{j_1 i_1} R_{j_2 i_2}, \\
\mathrm{Var}_2(R) &= \rho^{(2)}_{i_1 j_1, i_2 j_2} R_{j_1 i_1} R_{j_2 i_2} - \left(\rho^{(1)}_{i_1 j_1} R_{j_1 i_1}\right)^2, \\
\mathrm{Var}(R) &= \rho^{(1)}_{i_1 j_1} (R^2)_{j_1 i_1} - \left(\rho^{(1)}_{i_1 j_1} R_{j_1 i_1}\right)^2,
\end{aligned} \tag{8a}$$

with $R_{ji} = \langle j|R|i\rangle$. Clearly, other statistical properties of the ensemble are readily expressed in terms of the density tensor hierarchy. For example, the ensemble average of the product of the expectations of two different operators R and S is given by

$$\sum_\alpha w_\alpha \langle \psi_\alpha|R|\psi_\alpha\rangle \langle \psi_\alpha|S|\psi_\alpha\rangle = \rho^{(2)}_{i_1 j_1, i_2 j_2} R_{j_1 i_1} S_{j_2 i_2}, \tag{8b}$$

which can be used, together with information obtained from $\rho^{(1)}$, to calculate the covariance and correlation of R and S.

3. Isotropic spin-1/2 ensemble

As a simple example of the density tensor formalism, let us follow Breuer and Petruccione [1] and consider the case of an isotropic spin-1/2 ensemble. Let $\vec{v}$ be a vector in three dimensions, and consider the ensemble of spin-1/2 pure state density matrices

$$\rho(\vec{v}) = \tfrac{1}{2}(1 + \vec{v} \cdot \vec{\sigma}), \tag{9a}$$

with $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ the standard Pauli matrices and with a uniform probability distribution of $\vec{v}$ over the unit sphere $|\vec{v}| = 1$ specified by

$$w(\vec{v}) = \frac{1}{4\pi}\delta(|\vec{v}| - 1). \tag{9b}$$

(Clearly, $\vec{v}$ has the same significance as the label α used in the preceding section.) Defining

$$E[P(\vec{v})] = \int d^3v\, w(\vec{v}) P(\vec{v}), \tag{10a}$$

a standard calculation gives

$$E[1] = 1, \qquad E[v_s v_t] = \tfrac{1}{3}\delta_{st}, \ldots, \tag{10b}$$

with all averages of odd powers of $\vec{v}$ vanishing. From equation (9a), we have

$$\rho(\vec{v})_{ij} = \tfrac{1}{2}\big(\delta_{ij} + v_r \sigma^r_{ij}\big), \tag{11a}$$

and the general density tensor over this ensemble is defined by

$$\rho^{(n)}_{i_1 j_1, \ldots, i_n j_n} = E\big[\rho(\vec{v})_{i_1 j_1} \cdots \rho(\vec{v})_{i_n j_n}\big]. \tag{11b}$$

From equation (10b), the first three tensors in this hierarchy are now easily found to be

$$\rho^{(1)}_{i_1 j_1} = \tfrac{1}{2}\delta_{i_1 j_1},$$
$$\rho^{(2)}_{i_1 j_1, i_2 j_2} = \tfrac{1}{4}\big(\delta_{i_1 j_1}\delta_{i_2 j_2} + \tfrac{1}{3}\vec{\sigma}_{i_1 j_1} \cdot \vec{\sigma}_{i_2 j_2}\big), \tag{12}$$
$$\rho^{(3)}_{i_1 j_1, i_2 j_2, i_3 j_3} = \tfrac{1}{8}\big[\delta_{i_1 j_1}\delta_{i_2 j_2}\delta_{i_3 j_3} + \tfrac{1}{3}\big(\delta_{i_1 j_1}\vec{\sigma}_{i_2 j_2} \cdot \vec{\sigma}_{i_3 j_3} + \delta_{i_2 j_2}\vec{\sigma}_{i_1 j_1} \cdot \vec{\sigma}_{i_3 j_3} + \delta_{i_3 j_3}\vec{\sigma}_{i_1 j_1} \cdot \vec{\sigma}_{i_2 j_2}\big)\big].$$

Using the relations $\mathrm{Tr}\,\vec{\sigma} = 0$ and $(\vec{\sigma}^2)_{ij} = 3\delta_{ij}$, it is now easy to verify that the descent relations of equations (6a) and (6b) are satisfied by equation (12).

For the isotropic spin-1/2 ensemble, the generating function of equation (5d) becomes

$$G[a_{ij}] = E[e^{\rho(\vec{v})_{ij} a_{ij}}], \tag{13}$$

with $\rho(\vec{v})_{ij}$ given by equation (11a). Defining the vector $\vec{A}$ by

$$\vec{A} = \tfrac{1}{2}\vec{\sigma}_{ij} a_{ij}, \tag{14a}$$

a simple calculation gives

$$G[a_{ij}] = \exp\left(\frac{1}{2}\mathrm{Tr}\,a\right)\frac{\sinh|\vec{A}|}{|\vec{A}|} = \exp\left(\frac{1}{2}\mathrm{Tr}\,a\right)\left[1 + \frac{\vec{A}^2}{3!} + \frac{(\vec{A}^2)^2}{5!} + \cdots\right], \tag{14b}$$

from which one can read off the values of the low order density tensors given in equation (12). The verification of the descent relations of equation (7b) for the generating function of equation (14b) is given in appendix A.

4. Itô stochastic Schrödinger equation

We consider next a state vector $|\psi\rangle$ with a time evolution described by a stochastic Schrödinger equation, which is a frequently used model approximation to open system dynamics. In this case, the state vector and the corresponding pure state density matrix $\rho = |\psi\rangle\langle\psi|$ are implicit functions of the noise, which takes a different sequence of values for each history of the system. In the notation of section 2, the different histories are labelled by the subscript α, and the expectation of equation (4b) is an average over all possible histories. It is customary, however, in discussing stochastic Schrödinger equations to omit the subscript α, treating the history dependence of ρ as understood. So in this context, the definition of equation (4a) becomes

$$\rho^{(n)}_{i_1 j_1,\ldots,i_n j_n} = E\big[\rho_{i_1 j_1} \cdots \rho_{i_n j_n}\big], \tag{15}$$

with $E[\cdots]$ the stochastic expectation, and the generating function $G[a_{ij}]$ takes the same form as given in equation (5d), but with the subscript α omitted.

Our aim in this section is to derive an equation of motion for the generating function, which on expansion yields equations of motion for all density tensors $\rho^{(n)}$, taking as input the general pure state density matrix evolution constructed by Wiseman and Diósi [7], that corresponds to a given Lindblad form [8, 9] for the time evolution of the reduced density matrix $\rho^{(1)} = E[\rho]$. We begin by recapitulating the results of [7]. The most general evolution of a density matrix ρ that preserves $\mathrm{Tr}\,\rho = 1$ and obeys the complete positivity condition is the Lindblad form

$$\mathrm{d}\rho = \mathrm{d}t\,\mathcal{L}\rho, \tag{16a}$$

with

$$\mathcal{L}\rho \equiv -\mathrm{i}[H_{\mathrm{TOT}}, \rho] + c_k \rho c_k^\dagger - \tfrac{1}{2}\big\{c_k^\dagger c_k, \rho\big\}, \tag{16b}$$

with $\{,\}$ denoting the anticommutator and with the repeated index k summed. The set of Lindblad operators c_k describes the effects on the system of the reservoir or environment that is modelled by an external classical noise. Wiseman and Diósi show that the most general evolution of the pure state density matrix ρ for which $E[\mathrm{d}\rho]$ reduces to equations (16a) and (16b) takes the form

$$\mathrm{d}\rho = \mathrm{d}t\,\mathcal{L}\rho + |\mathrm{d}\phi\rangle\langle\psi| + |\psi\rangle\langle\mathrm{d}\phi|. \tag{17a}$$

Here $|\mathrm{d}\phi\rangle$ is a state vector that is a pure noise term, so that

$$E[|\mathrm{d}\phi\rangle] = 0, \tag{17b}$$

that is orthogonal to $|\psi\rangle$, so that

$$\langle\psi|\mathrm{d}\phi\rangle = 0, \tag{17c}$$

and that obeys

$$|\mathrm{d}\phi\rangle\langle\mathrm{d}\phi| = \mathrm{d}t\,W. \tag{17d}$$

The operator W is the Diósi transition rate operator [5] given by

$$\begin{aligned} W &= \mathcal{L}\rho - \{\rho, \mathcal{L}\rho\} + \rho\,\mathrm{Tr}(\rho\mathcal{L}\rho) \\ &= (c_k - \langle c_k\rangle)\rho(c_k - \langle c_k\rangle)^\dagger, \end{aligned} \tag{18}$$

where $\langle c_k\rangle$ is a shorthand for the quantum state expectation $\langle\psi|c_k|\psi\rangle = \mathrm{Tr}\,\rho c_k$. Although $|\mathrm{d}\phi\rangle\langle\mathrm{d}\phi|$ is completely fixed, Wiseman and Diósi show that $|\mathrm{d}\phi\rangle|\mathrm{d}\phi\rangle$ is free, with different choices for this and different phase choices for c_k corresponding to different pure state

evolutions (or 'unravellings') that yield the same evolution of equations (16a) and (16b) for the reduced density matrix ρ.

Wiseman and Diósi further show that $|\mathrm{d}\phi\rangle$ can be parameterized by complex Wiener processes by writing

$$|\mathrm{d}\phi\rangle = (c_k - \langle c_k \rangle)|\psi\rangle \,\mathrm{d}\xi_k^*, \tag{19a}$$

with

$$E[\mathrm{d}\xi_k] = E[\mathrm{d}\xi_k^*] = 0 \tag{19b}$$

and with

$$\begin{aligned} \mathrm{d}\xi_j(t)\,\mathrm{d}\xi_k^*(t) &= \mathrm{d}t\,\delta_{jk} \\ \mathrm{d}\xi_j(t)\,\mathrm{d}\xi_k(t) &= \mathrm{d}t\,u_{jk}, \end{aligned} \tag{19c}$$

where $u_{kj} = u_{jk}$ is a set of arbitrary complex numbers subject to the condition that the norm of the complex matrix $\mathbf{u} \equiv [u_{jk}]$ be less than or equal to 1 (see equations (4.10) and (4.11) of [7]). In terms of this parameterization of $|\mathrm{d}\phi\rangle$, the pure state evolution of equation (17a) takes the form

$$\mathrm{d}\rho = \mathrm{d}t\,\mathcal{L}\rho + (c_k - \langle c_k \rangle)\rho\,\mathrm{d}\xi_k^* + \rho(c_k - \langle c_k \rangle)^\dagger\,\mathrm{d}\xi_k, \tag{19d}$$

and the corresponding stochastic Schrödinger equation for the wavefunction is [7]

$$\begin{aligned} \mathrm{d}|\psi\rangle &= -\mathrm{i}H_\psi\,\mathrm{d}t|\psi\rangle + (c_k - \langle c_k \rangle)\,\mathrm{d}\xi_k^*|\psi\rangle, \\ -\mathrm{i}H_\psi &= -\mathrm{i}H_{\mathrm{TOT}} - \tfrac{1}{2}\big(c_k^\dagger c_k - 2\langle c_k \rangle^* c_k + \langle c_k \rangle^* \langle c_k \rangle\big). \end{aligned} \tag{19e}$$

We proceed now to use pure state evolution of equation (19d) to calculate the evolution equation for the generating function

$$G[a_{ij}] = E[\exp(\rho_{ij}a_{ij})]. \tag{20a}$$

To calculate the differential of equation (20a), we use the Itô stochastic calculus rule for the differential of a function $f(w)$ of a stochastic variable w,

$$\mathrm{d}f(w) = \mathrm{d}w\,f'(w) + \tfrac{1}{2}(\mathrm{d}w)^2 f''(w). \tag{20b}$$

Applying this to equation (20a), we get

$$\mathrm{d}G[a_{ij}] = E\big[\big(\mathrm{d}\rho_{mr}a_{mr} + \tfrac{1}{2}\,\mathrm{d}\rho_{mr}a_{mr}\,\mathrm{d}\rho_{pq}a_{pq}\big)\exp(\rho_{ij}a_{ij})\big]. \tag{20c}$$

Substituting equation (19d) for $\mathrm{d}\rho$, and using equations (19a)–(19c), together with the Itô calculus rule $E[\mathrm{d}wf(w)] = 0$, we get

$$\mathrm{d}G[a_{ij}] = \mathrm{d}t\,E\big[\big(a_{mr}(\mathcal{L}\rho)_{mr} + \tfrac{1}{2}a_{mr}a_{pq}C_{mr,pq}\big)\exp(\rho_{ij}a_{ij})\big], \tag{21a}$$

with the coefficient of the quadratic term in a_{ij} given by

$$\begin{aligned} C_{mr,pq} = C_{pq,mr} &= \mathrm{d}\rho_{mr}\mathrm{d}\rho_{pq} \\ &= \langle m|(c_k - \langle c_k \rangle)\rho|r\rangle\langle p|\rho(c_k - \langle c_k \rangle)^\dagger|q\rangle \\ &\quad + \langle m|\rho(c_k - \langle c_k \rangle)^\dagger|r\rangle\langle p|(c_k - \langle c_k \rangle)\rho|q\rangle \\ &\quad + \langle m|(c_k - \langle c_k \rangle)\rho|r\rangle\langle p|(c_\ell - \langle c_\ell \rangle)\rho|q\rangle u_{k\ell}^* \\ &\quad + \langle m|\rho(c_k - \langle c_k \rangle)^\dagger|r\rangle\langle p|\rho(c_\ell - \langle c_\ell \rangle)^\dagger|q\rangle u_{k\ell}. \end{aligned} \tag{21b}$$

This expression can be rearranged by using the identity, valid for general operators A, B, general states $|r\rangle$, $|m\rangle$ and general pure state (idempotent) density matrix ρ,

$$\rho A|r\rangle\langle m|B\rho = \rho\langle m|B\rho A|r\rangle, \tag{22a}$$

giving an alternative result for $C_{mr,pq}$

$$C_{mr,pq} = W_{mq}\rho_{pr} + W_{pr}\rho_{mq} + [(c_k - \langle c_k \rangle)\rho]_{mq} u_{k\ell}^* [(c_\ell - \langle c_\ell \rangle)\rho]_{pr}$$
$$+ [\rho(c_k - \langle c_k \rangle)^\dagger]_{pr} u_{k\ell}[\rho(c_\ell - \langle c_\ell \rangle)^\dagger]_{mq}, \tag{22b}$$

where we have used equation (18) defining the operator W, and where we use the subscript notation of equation (3b) for matrix elements, so that in general $A_{mr} = \langle m|A|r \rangle$.

From the evolution equation of equations (21a), (21b) and (22b) for the generating function, by expansion in powers of a we can read off the evolution equation for the general density tensor of order n. Employing now the condensed notation of equation (5e), in which matrix indices are not indicated explicitly, we have

$$d\rho^{(n)} = dt\, E\left[\sum_{\ell=1}^{n} (\rho_1 \cdots \rho_n)_\ell (\mathcal{L}\rho)_\ell + \sum_{\ell<m=1}^{n} (\rho_1 \cdots \rho_n)_{\ell m} C_{\ell m}\right]. \tag{23a}$$

Here $(\rho_1 \cdots \rho_n)_\ell$ denotes the product $\prod_{j=1}^{n} \rho_j$ with the factor ρ_ℓ omitted and, similarly, $(\rho_1 \cdots \rho_n)_{\ell m}$ denotes the product $\prod_{j=1}^{n} \rho_j$ with the factors ρ_ℓ and ρ_m omitted[2]. The coefficient $C_{\ell m}$ is given by

$$C_{\ell m} = C_{m\ell} = [(c_k - \langle c_k \rangle)\rho]_\ell [\rho(c_k - \langle c_k \rangle)^\dagger]_m + [\rho(c_k - \langle c_k \rangle)^\dagger]_\ell [(c_k - \langle c_k \rangle)\rho]_m$$
$$+ [(c_k - \langle c_k \rangle)\rho]_\ell [(c_{\bar{k}} - \langle c_{\bar{k}} \rangle)\rho]_m u_{k\bar{k}}^* + [\rho(c_k - \langle c_k \rangle)^\dagger]_\ell [\rho(c_{\bar{k}} - \langle c_{\bar{k}} \rangle)^\dagger]_m u_{k\bar{k}}, \tag{23b}$$

which corresponds in an obvious way to equation (21b) when matrix elements are written explicitly between states $\langle m|$ and $|r \rangle$ in the Hilbert space labelled by ℓ and between states $\langle p|$ and $|q \rangle$ in the Hilbert space labelled by m. (No relation is implied between m used as a state label and m used as a Hilbert space label.) Since $C_{\ell m}$ in equation (23a), which depends through the terms involving $u_{k\bar{k}}$ on the choice of unravelling, is multiplied by two powers of a, it does not contribute to the evolution equation for the reduced density matrix $\rho^{(1)}$. So as expected, the reduced density matrix evolution is given solely by the Lindblad term and is independent of the choice of unravelling. Higher density tensors $\rho^{(n)}$, with $n \geqslant 2$, have evolution equations that receive contributions from $C_{\ell m}$, and so contain information that distinguishes between different unravellings of the Lindblad evolution.

As a simple illustration of how the tensors $\rho^{(n)}$ for $n \geqslant 2$ distinguish between different unravellings, let us consider the case of real noise, $d\xi_k = d\xi_k^*$, for which $u_{jk} = \delta_{jk}$, and with a single Lindblad c_1, which we choose as either $c_1 = A$ or $c_1 = iA$, with A a self-adjoint operator. Both choices of c_1 lead to the same Lindblad, since $\mathcal{L}$ is invariant under rephasing of c_k, but through the $u_{k\bar{k}}$ terms they lead to different expressions for $C_{mr,pq}$. When $c_k = A$, we find from equation (21b)

$$C_{mr,pq} = \langle m|\{A - \langle A \rangle, \rho\}|r \rangle \langle p|\{A - \langle A \rangle, \rho\}|q \rangle$$
$$= \langle m|[\rho, [\rho, A]]|r \rangle \langle p|[\rho, [\rho, A]]|q \rangle, \tag{24a}$$

while when $c_k = iA$, we have instead

$$C_{mr,pq} = -\langle m|[A, \rho]|r \rangle \langle p|[A, \rho]|q \rangle. \tag{24b}$$

We will return to this example in section 10.

Using the expression of equation (21a) for the time evolution of the generating function, the descent equations of equation (7b) can be verified; this calculation is carried out in appendix B.

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[2] For $n = 1$, $(\rho_1)_1 = 1$ and $(\rho_1)_{\ell m} = 0$, while for $n = 2$, $(\rho_1\rho_2)_{12} = 1$.

5. Jump process Schrödinger equation

As our next density tensor application we consider the jump process (piecewise deterministic process or PDP) Schrödinger equation, given by

$$d|\psi\rangle = A\, dt|\psi\rangle + B_k\, dN_k|\psi\rangle, \tag{25a}$$

where a sum over k is understood, with A and the B_k general (non-self-adjoint) operators, and with the dN_k independent discrete random variables obeying

$$dN_j\, dN_k = \delta_{jk}\, dN_k, \qquad dN_j\, dt = 0. \tag{25b}$$

Straightforward calculation shows that this process preserves the norm of $|\psi\rangle$ and the pure state condition $\rho^2 = \rho = |\psi\rangle\langle\psi|$, provided that A and B obey the restrictions

$$\begin{aligned}
\langle A + A^\dagger \rangle &= 0, \\
\langle B_k + B_k^\dagger + B_k^\dagger B_k \rangle &= 0,
\end{aligned} \tag{25c}$$

with no summation over k on the second line, which must hold individually for each value of k. Corresponding to equation (25a), the density matrix obeys the evolution equation

$$d\rho = (A\rho + \rho A^\dagger)\, dt + Q_k\, dN_k, \qquad Q_k = B_k\rho + \rho B_k^\dagger + B_k\rho B_k^\dagger, \tag{25d}$$

with a sum over k understood in the dN_k term in the first equation, but no sum over k understood in the second equation.

Let now $E_{|\psi\rangle}[\cdots]$ denote an expectation conditioned on the current value of the wavefunction being $|\psi\rangle$, and $E[\cdots]$ be the expectation value over the entire history of the jump process (which leads to an ensemble of different current values of the wavefunction). We wish to find restrictions on A, B_k and on

$$E_{|\psi\rangle}[dN_k] \equiv v_k\, dt, \tag{26a}$$

such that the expectation of $d\rho$ takes the Lindblad form of equation (16b), that is,

$$\begin{aligned}
E[d\rho] &= dt\, \mathcal{L}\rho \\
\mathcal{L}\rho &= -i[H_{\text{TOT}}, \rho] + c_k\rho c_k^\dagger - \tfrac{1}{2}\{c_k^\dagger c_k, \rho\}.
\end{aligned} \tag{26b}$$

Making the ansatz

$$B_k = \frac{c_k - K_k}{v_k^{\frac{1}{2}}} - 1, \tag{27a}$$

with K_k constants (this ansatz includes both the standard quantum jump equation ($K_k = 0$) and the orthogonal jump equation ($K_k = \langle c_k \rangle$), as special cases; see Schack and Brun [10] for a concise review), some calculation shows that the conditions of equations (25c) and (26b) are satisfied if we choose

$$\begin{aligned}
v_k &= \langle (c_k - K_k)^\dagger (c_k - K_k) \rangle, \\
A &= -iH_{\text{TOT}} - \tfrac{1}{2}c_k^\dagger c_k + \tfrac{1}{2}\langle c_k^\dagger c_k \rangle + c_k K_k^* - \tfrac{1}{2}(\langle c_k \rangle K_k^* + \langle c_k \rangle^* K_k).
\end{aligned} \tag{27b}$$

Let us now define the order n density tensor for the jump models by

$$\rho^{(n)} = E\left[\prod_{\ell=1}^{n} \rho_\ell\right], \tag{28a}$$

where we use the condensed notation of equation (5*e*). For the differential of this, we find

$$
\mathrm{d}\rho^{(n)} = E\Bigg[\sum_{\ell=1}^{n}(\rho_1\cdots\rho_n)_\ell\,\mathrm{d}\rho_\ell + \sum_{\ell<m=1}^{n}(\rho_1\cdots\rho_n)_{\ell m}\,\mathrm{d}\rho_\ell\,\mathrm{d}\rho_m
$$
$$
+ \sum_{\ell<m<p=1}^{n}(\rho_1\cdots\rho_n)_{\ell m p}\,\mathrm{d}\rho_\ell\,\mathrm{d}\rho_m\,\mathrm{d}\rho_p + \cdots + \mathrm{d}\rho_1\,\mathrm{d}\rho_2\,\mathrm{d}\rho_3\cdots\mathrm{d}\rho_{n-1}\,\mathrm{d}\rho_n\Bigg],
$$
$$(28b)$$

where all powers of $\mathrm{d}\rho$ must be retained because $\mathrm{d}N_k^2 = \mathrm{d}N_k$. Using the conditional probability formula $p(|\psi\rangle \cap \mathrm{d}N_k) = p(\mathrm{d}N_k||\psi\rangle)p(|\psi\rangle)$, we get the conditional expectation formula, valid for an arbitrary function F of the state $|\psi\rangle$,

$$
E[F(|\psi\rangle)\,\mathrm{d}N_k] = E[F(|\psi\rangle)E_{|\psi\rangle}[\mathrm{d}N_k]] = E[F(|\psi\rangle)v_k]. \qquad (29a)
$$

Using this equation to evaluate the higher order terms in equation (28*b*), together with equation (26*b*) for the leading term, we get

$$
\mathrm{d}\rho^{(n)} = \mathrm{d}t\,E\Bigg[\sum_{\ell=1}^{n}(\rho_1\cdots\rho_n)_\ell(\mathcal{L}\rho)_\ell + \sum_{\ell<m=1}^{n}(\rho_1\cdots\rho_n)_{\ell m}v_k(Q_k)_\ell(Q_k)_m
$$
$$
+ \sum_{\ell<m<p=1}^{n}(\rho_1\cdots\rho_n)_{\ell m p}v_k(Q_k)_\ell(Q_k)_m(Q_k)_p + \cdots
$$
$$
+ v_k(Q_k)_1(Q_k)_2(Q_k)_3\cdots(Q_k)_{n-1}(Q_k)_n\Bigg], \qquad (29b)
$$

with a sum over k in each term containing v_k.

Writing the corresponding generating function in compact notation as

$$
G[a] = E[\mathrm{e}^{a\cdot\rho}], \qquad (30a)
$$

the evolution equation for G is given, with the k sum now indicated explicitly, by

$$
\mathrm{d}G[a] = E[\mathrm{e}^{a\cdot\mathrm{d}\rho}\,\mathrm{e}^{a\cdot\rho}] - E[\mathrm{e}^{a\cdot\rho}]
$$
$$
= E\Bigg[\left(\sum_{p=1}^{\infty}\frac{(a\cdot\mathrm{d}\rho)^p}{p!} \right) \mathrm{e}^{a\cdot\rho} \Bigg]
$$
$$
= \mathrm{d}t\,E\Bigg[\left(a\cdot\mathcal{L}\rho + \sum_{p=2}^{\infty}\sum_{k}v_k\frac{(a\cdot Q_k)^p}{p!} \right) \mathrm{e}^{a\cdot\rho} \Bigg]. \qquad (30b)
$$

From equation (30*b*) and the identities (which follow, after some algebra, from equations (16*b*), (25*d*), (27*a*) and (27*b*))

$$
\{\rho, \mathcal{L}\rho\} = \mathcal{L}\rho - \sum_{k}v_k Q_k^2,
$$
$$
\{\rho, Q_k\} = Q_k - Q_k^2, \qquad (30c)
$$

one can prove that equation (30*b*) obeys the descent equations, as shown in appendix C.

6. The density tensor for quantum noise and its kinematical properties

Let us now consider a closed quantum system, consisting of a system $\mathcal{S}$ interacting with an environment $\mathcal{E}$. In such a situation, one does not have a classical probability distribution

w_α and fluctuations associated with this probability distribution. Instead, one deals with the system plus environment as the only pure state that is given, with the fluctuations that are averaged over in deriving the master equation coming from quantum fluctuations associated with the system–environment interaction. Weighted averages of the sort that we have used in our definition of equation (4a) appear only when the total state is a mixture of pure states, such as a thermal state, but in this case important system quantum fluctuations still occur in each pure state component of this mixture. In order to describe this more general situation, we shall have to generalize our definition of a density tensor hierarchy.

To achieve this, we initially suppose the overall system plus environment to have the pure state density matrix ρ. We denote the system basis states by $|i\rangle$, as well as $|j\rangle$, and denote the environment basis states by $|e_a\rangle$, $a = 1, 2, \ldots$. A general density matrix element has the form $\langle e_1 i|\rho|e_2 j\rangle$, and the standard reduced density matrix, with the environment traced out, is defined by

$$\rho_{ij}^{(1)} = (\mathrm{Tr}_{\mathcal{E}}\rho)_{ij} = \sum_e \langle ei|\rho|ej\rangle. \tag{31}$$

In order to recapture fluctuations that are averaged over in the trace in equation (31), we define the density tensor $\rho^{(n)}$ by

$$\rho_{i_1 j_1, i_2 j_2, \ldots, i_n j_n}^{(n)} = \sum_{e_1, e_2, \ldots, e_n} \langle e_1 i_1|\rho|e_2 j_1\rangle\langle e_2 i_2|\rho|e_3 j_2\rangle \cdots \langle e_{n-1} i_{n-1}|\rho|e_n j_{n-1}\rangle\langle e_n i_n|\rho|e_1 j_n\rangle$$
$$= \mathrm{Tr}_{\mathcal{E}}\rho_{i_1 j_1}\rho_{i_2 j_2} \cdots \rho_{i_n j_n}. \tag{32a}$$

Here we have defined $\rho_{i_\ell j_\ell}$ as the matrix, labelled by the system state labels i_ℓ, j_ℓ, acting on the environment Hilbert space $\mathcal{H}_{\mathcal{E}}$ according to

$$(\rho_{i_\ell j_\ell})_{e_1 e_2} = \langle e_1|\rho_{i_\ell j_\ell}|e_2\rangle = \langle e_1 i_\ell|\rho|e_2 j_\ell\rangle. \tag{32b}$$

The density tensor $\rho^{(n)}$ is again an operator on a tensor product of system Hilbert spaces $\prod_{\ell=1}^n \mathcal{H}_{\mathcal{S};\ell}$. Thus, in a condensed notation analogous to that of equation (5e), we can also write equation (32a) as

$$\rho^{(n)} = \mathrm{Tr}_{\mathcal{E}}\rho_1\rho_2 \cdots \rho_n, \tag{32c}$$

where ρ_ℓ is an operator acting on $\mathcal{H}_{\mathcal{E}} \otimes \mathcal{H}_{\mathcal{S};\ell}$.

We have avoided using a product notation $\prod_{\ell=1}^n$ in equation (32c) because the factors $\rho_{i_\ell j_\ell}$ in equation (32a) and ρ_ℓ in equation (32c) are different operators on the environment for each ℓ and thus do not commute. Hence, the density tensor is not symmetric under permutation of its pair indices $i_\ell j_\ell$, but it is symmetric under cyclical permutation of the indices, as a result of the cyclic symmetry of the trace. For $n = 2$, cyclic symmetry is equivalent to symmetry under pair index interchange, and for $n = 3$, using the identity

$$\mathrm{Tr}\,ABC = \mathrm{Tr}\,\tfrac{1}{2}([A, B]C + \{A, B\}C), \tag{33}$$

cyclic symmetry is equivalent to the statement that the density tensor $\rho^{(3)}$ can be written as the sum of two tensors $\rho^{(3)} = \rho^{(3S)} + \rho^{(3A)}$, with $\rho^{(3S)}$ completely symmetric, and $\rho^{(3A)}$ completely antisymmetric, under pair index interchange. Also because the density tensor is not totally symmetric in its pair indices, we cannot introduce a generating function by imitating equation (5d).

Similarly, because of factor non-commutativity, the density tensor satisfies only a subset of the descent equations of equations (6a), (6b) and (7b). Contraction with $\delta_{i_\ell j_\ell}$ does not lead to a descent condition, since $\delta_{i_\ell j_\ell}\rho_{i_\ell j_\ell}$ is not unity, but rather $\mathrm{Tr}_{\mathrm{S}}\rho$, the reduced density matrix that acts on the environment when the system is traced out. Contraction of a general j_k with a general i_ℓ for $k \neq \ell$ gives nothing useful, since in general non-commuting factors stand

between ρ_k and ρ_ℓ. However, when a column index j_k is contracted with the adjacent row index i_{k+1}, the two density matrices to which they are attached are linked to form the product $\rho^2 = \rho$, and so we get the descent relation of equation (6b), and others related to it by cyclic permutation symmetry,

$$\delta_{j_1 i_2} \rho^{(n)}_{i_1 j_1, i_2 j_2, \ldots, i_n j_n} = \rho^{(n-1)}_{i_1 j_2, i_3 j_3, \ldots, i_n j_n}. \tag{34}$$

As noted before, even when $\rho^2 \neq \rho$, the descent relation corresponding to equation (34) is still useful for checking calculations. Since we cannot define a generating function as in equation (5d), in the quantum noise case we do not have analogues of the descent equations in the form of equation (7b); when verifying the descent equations in the various cases considered below, we will work directly from equation (34).

We will also consider a more general definition of the density tensor, corresponding to the case in which the system plus environment is in a mixed state composed of pure states ρ_α with weights w_α. Typically, α refers to an eigenvalue of a conserved quantum number of the total system, such as the energy; when the environment is considered in the independent particle approximation, with the system back reaction on the environment neglected, α then can refer to the energies and momenta of each environmental particle. In this case, we define the density tensor by

$$\rho^{(n)}_{i_1 j_1, i_2 j_2, \ldots, i_n j_n} = \sum_\alpha w_\alpha \rho^{(n)}_{\alpha; i_1 j_1, i_2 j_2, \ldots, i_n j_n}, \tag{35a}$$

with

$$\rho^{(n)}_{\alpha; i_1 j_1, i_2 j_2, \ldots, i_n j_n} = \mathrm{Tr}_{\mathcal{E}} \rho_{\alpha; i_1 j_1} \rho_{\alpha; i_2 j_2} \cdots \rho_{\alpha; i_n j_n}. \tag{35b}$$

This definition gives information about both the quantum noise or fluctuations contained within each ρ_α, and the classical noise or fluctuations associated with the probability distribution w_α. Note that in the mixed state case one could also define a density tensor that is a direct analogue of the classical noise definition of section 2 by

$$\rho^{(n);\mathrm{CL}}_{i_1 j_1, i_2 j_2, \ldots, i_n j_n} = \sum_\alpha w_\alpha \prod_{\ell=1}^{n} \mathrm{Tr}_{\mathcal{E}} \rho_{\alpha; i_\ell j_\ell} \tag{36}$$

which would give information only about the classical noise fluctuations associated with the probability distribution w_α. In the examples computed in the following sections, where a weak coupling approximation is made, the definition of equation (36) typically contains no more information than could be gotten from a product of n reduced density matrix factors, each of the form $\sum_\alpha w_\alpha \mathrm{Tr}_{\mathcal{E}} \rho_{\alpha; i_\ell j_\ell}$.

As already noted, the density tensor $\rho^{(n)}$ is not measurable by any operation on the system Hilbert space. Its construction requires knowledge of the full system plus environment density matrix, which is not experimentally accessible for complex environments. Nonetheless $\rho^{(n)}$ is computable in any theory of the system–environment interaction, and we believe it to be of conceptual and theoretical interest, even if not of direct empirical relevance.

We close out this section by noting that in the quantum noise case there is no analogue of equation (8a), which relates the positive semidefinite variations $\mathrm{Var}_{1,2}$ to the density tensor $\rho^{(2)}$ in the classical noise case. The closest analogue we find to the fluctuation formulae of equation (8a) involves the $n = 3$ density tensor. The reason for this is that whereas $E[1] = 1$, the trace over the environment of unity is the dimension of the environmental Hilbert space; to get a unit trace over the environment we must include a factor of $\rho_{\mathcal{E}} \equiv \mathrm{Tr}_{\mathcal{S}} \rho$, the reduced density matrix for the environment. This pushes up the order of the density tensor involved from 2 to 3. Specifically, let $A_{\mathcal{S}}$ be an operator acting on $\mathcal{H}_{\mathcal{S}}$, but which acts as the

unit operator on $\mathcal{H}_\mathcal{E}$. In place of the expectations used in the classical noise discussion of equations (1a) through (2d), in the quantum noise case of system plus environment we consider the expression $A_\mathcal{E} \equiv \mathrm{Tr}_\mathcal{S}\rho A_\mathcal{S}$, which is an operator on the environmental Hilbert space. The trace of this operator over the environment is $\mathrm{Tr}_\mathcal{E} A_\mathcal{E} = \mathrm{Tr}_\mathcal{E}\mathrm{Tr}_\mathcal{S}\,\rho A_\mathcal{S} = \mathrm{Tr}_\mathcal{S}(\mathrm{Tr}_\mathcal{E}\rho)A_\mathcal{S} = \mathrm{Tr}_\mathcal{S}\rho^{(1)}A_\mathcal{S}$, giving the expectation of the operator $A_\mathcal{S}$ when the environment is not observed. On the other hand, the expectation of this operator formed from the environmental reduced density matrix is $\mathrm{Tr}_\mathcal{E}\rho_\mathcal{E} A_\mathcal{E}$. The mean squared fluctuation of this operator over the environment is positive semidefinite, and is given by

$$\mathrm{Tr}_\mathcal{E}\rho_\mathcal{E}(A_\mathcal{E} - \mathrm{Tr}_\mathcal{E}\rho_\mathcal{E} A_\mathcal{E})^2 = \mathrm{Tr}_\mathcal{E}\rho_\mathcal{E} A_\mathcal{E}^2 - (\mathrm{Tr}_\mathcal{E}\rho_\mathcal{E} A_\mathcal{E})^2, \tag{37a}$$

where we have used the fact that $\mathrm{Tr}_\mathcal{E}\rho_\mathcal{E} = \mathrm{Tr}\,\rho = 1$. Reexpressing equation (37a) entirely in terms of the pure state density matrix ρ, we have

$$\mathrm{Tr}_\mathcal{E}\mathrm{Tr}_\mathcal{S}\rho(\mathrm{Tr}_\mathcal{S}\rho A_\mathcal{S})^2 - (\mathrm{Tr}_\mathcal{E}\mathrm{Tr}_\mathcal{S}\rho\,\mathrm{Tr}_\mathcal{S}\rho A_\mathcal{S})^2 = \delta_{j_1 i_1} A_{\mathcal{S} j_2 i_2} A_{\mathcal{S} j_3 i_3} \rho^{(3S)}_{i_1 j_1, i_2 j_2, i_3 j_3}$$
$$- \left(\delta_{j_1 i_1} A_{\mathcal{S} j_2 i_2} \rho^{(2)}_{i_1 j_1, i_2 j_2}\right)^2, \tag{37b}$$

where we have used the fact that the right-hand side of equation (37b) involves only the symmetric part of the order 3 density tensor. Thus, as noted above, where an $n = 2$ density tensor appears in equation (8a), an $n = 3$ density tensor appears in equation (37b), and where an $n = 1$ density tensor appears in equation (8a), an $n = 2$ density tensor appears in equation (37b).

7. Collisional Brownian motion

As our first application of equations (32a)–(32c) and equations (35a), (35b), we consider the collisional Brownian motion of a massive Brownian particle immersed in a bath of scattering particles. We work in the approximation of neglecting recoil of the Brownian particle and of treating the bath as a collection of free particles of mass m. We consider the pure state density matrix corresponding to definite momenta $\{\vec{k}_i\}$ of the bath particles, calculate the corresponding order n density tensor defined by equations (32a)–(32c) and then average over the thermal distribution of the bath particles as in equations (35a), (35b). Thus, the initial density matrix for the total system, corresponding to the factor ρ_ℓ in equation (33d), is

$$\rho_\ell^{\mathrm{TOT}} = \rho_\ell \rho_\mathcal{E}, \tag{38a}$$

with ρ_ℓ the initial density matrix of the Brownian particle, characterized by its coordinate matrix elements $\langle\vec{R}_\ell|\rho_\ell|\vec{R}'_\ell\rangle$, and with $\rho_\mathcal{E}$ the product density matrix for the bath particles,

$$\rho_\mathcal{E} = \prod_i |\vec{k}_i\rangle\langle\vec{k}_i|. \tag{38b}$$

Since the bath particle scatterings are all independent, we focus on the effect of the scattering of a single bath particle, of initial momentum $\vec{k}$, on the Brownian particle, which we take to be in a superposition of position eigenstates. Thus, the initial state of the Brownian particle and the bath particle that we are considering is

$$|I\rangle = \sum_{\vec{R}} c_{\vec{R}}|\vec{R}\rangle|\vec{k}\rangle, \tag{39a}$$

corresponding to an initial state density matrix

$$\rho_I = |I\rangle\langle I|$$
$$= \sum_{\vec{R}} \sum_{\vec{R}'} c_{\vec{R}} c_{\vec{R}'}^* |\vec{R}\rangle|\vec{k}\rangle\langle\vec{k}|\langle\vec{R}'|. \tag{39b}$$

The corresponding Brownian particle matrix element of ρ_I, which is still an operator on the bath particle state, takes the form

$$\langle \vec{R} | \rho_I | \vec{R}' \rangle = \rho(\vec{R}, \vec{R}') |\vec{k}\rangle \langle \vec{k}|, \tag{39c}$$

with

$$\rho(\vec{R}, \vec{R}') = c_{\vec{R}} c_{\vec{R}'}^*. \tag{39d}$$

Asymptotically, the effect of the scattering is to replace the initial state $|I\rangle$ by $|F\rangle = S|I\rangle$, with S the scattering matrix. Substituting equation (39a), and using translation invariance to relate the scattering matrix S with the Brownian particle at a general coordinate, to the scattering matrix S_0 with the Brownian particle at the origin, we get [11]

$$\begin{aligned}
|F\rangle = S|I\rangle &= \sum_{\vec{R}} c_{\vec{R}} S |\vec{R}\rangle |\vec{k}\rangle \\
&= \sum_{\vec{R}} c_{\vec{R}} |\vec{R}\rangle \, \mathrm{e}^{-\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}} S_0 \, \mathrm{e}^{\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}} |\vec{k}\rangle,
\end{aligned} \tag{40a}$$

with $\vec{k}_{\mathrm{OP}}$ the momentum operator for the bath particle. The corresponding final density matrix is then

$$\begin{aligned}
\rho_F = |F\rangle\langle F| &= \sum_{\vec{R}} \sum_{\vec{R}'} c_{\vec{R}} c_{\vec{R}'}^* |\vec{R}\rangle \, \mathrm{e}^{-\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}} S_0 \, \mathrm{e}^{\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}} |\vec{k}\rangle \langle \vec{k}| \, \mathrm{e}^{-\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}'} S_0^\dagger \, \mathrm{e}^{\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}'} \langle \vec{R}'|, \tag{40b}
\end{aligned}$$

and the Brownian particle matrix element of ρ_F, which is again an operator acting on the bath particle, is

$$\langle \vec{R} | \rho_F | \vec{R}' \rangle = \rho(\vec{R}, \vec{R}') \, \mathrm{e}^{-\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}} S_0 \, \mathrm{e}^{\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}} |\vec{k}\rangle \langle \vec{k}| \, \mathrm{e}^{-\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}'} S_0^\dagger \, \mathrm{e}^{\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}'}. \tag{40c}$$

Substituting this expression into equation (33d), we get

$$\begin{aligned}
\rho^{(n)}_{\vec{R}_1 \vec{R}'_1, \ldots, \vec{R}_n \vec{R}'_n; F} &= \prod_{\ell=1}^{n} \rho(\vec{R}_\ell, \vec{R}'_\ell) \langle \vec{k}| \, \mathrm{e}^{-\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}'_\ell} S_0^\dagger \, \mathrm{e}^{\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}'_\ell} \, \mathrm{e}^{-\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}_{\ell+1}} S_0 \, \mathrm{e}^{\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot \vec{R}_{\ell+1}} |\vec{k}\rangle \\
&= \prod_{\ell=1}^{n} \rho(\vec{R}_\ell, \vec{R}'_\ell) \langle \vec{k}| S_0^\dagger \, \mathrm{e}^{\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot (\vec{R}'_\ell - \vec{R}_{\ell+1})} S_0 |\vec{k}\rangle \, \mathrm{e}^{\mathrm{i}\vec{k} \cdot (\vec{R}_{\ell+1} - \vec{R}'_\ell)}, \tag{40d}
\end{aligned}$$

with $\vec{R}_{n+1} = \vec{R}_1$. The matrix element appearing in the final line of equation (40d) is one that is familiar from the standard calculation of the reduced density matrix (that is $\rho^{(1)}_{\vec{R}, \vec{R}'}$) for collisional decoherence [12]. Writing

$$\langle \vec{k}| S_0^\dagger \, \mathrm{e}^{\mathrm{i}\vec{k}_{\mathrm{OP}} \cdot (\vec{R}'_\ell - \vec{R}_{\ell+1})} S_0 |\vec{k}\rangle \, \mathrm{e}^{\mathrm{i}\vec{k} \cdot (\vec{R}_{\ell+1} - \vec{R}'_\ell)} = 1 + f(\vec{R}_{\ell+1} - \vec{R}'_\ell), \tag{41a}$$

with f proportional to the square of the scattering amplitude, the product of matrix elements in equation (40d) can be written, to second-order accuracy in the scattering amplitude, as

$$\prod_{\ell=1}^{n} [1 + f(\vec{R}_{\ell+1} - \vec{R}'_\ell)] \simeq 1 + \sum_{\ell=1}^{n} f(\vec{R}_{\ell+1} - \vec{R}'_\ell). \tag{41b}$$

We also note that equation (39c), when substituted into equation (32c), implies that the value of $\rho^{(n)}$ before the scattering is

$$\rho^{(n)}_{\vec{R}_1 \vec{R}'_1, \ldots, \vec{R}_n \vec{R}'_n; I} = \prod_{\ell=1}^{n} \rho(\vec{R}_\ell, \vec{R}'_\ell). \tag{41c}$$

Thus when the approximation of equation (41b) is substituted into equation (40d), we get

$$\rho^{(n)}_{\vec{R}_1 \vec{R}'_1, \ldots, \vec{R}_n \vec{R}'_n; F} - \rho^{(n)}_{\vec{R}_1 \vec{R}'_1, \ldots, \vec{R}_n \vec{R}'_n; I} = \left[\sum_{\ell=1}^{n} f(\vec{R}_{\ell+1} - \vec{R}'_\ell) \right] \rho^{(n)}_{\vec{R}_1 \vec{R}'_1, \ldots, \vec{R}_n \vec{R}'_n; I}. \quad (41d)$$

At this point our work is essentially finished, since the remaining steps are identical to the standard calculation [11–13] proceeding from the $n = 1$ case of equation (41d), and the structure of equation (41d) makes it clear how to generalize the standard result for $\rho^{(1)}$ to the case of general $\rho^{(n)}$. In brief, the standard procedure is to multiply the right-hand side of equation (41d) by the number of scattering particles, which combines with a normalizing factor of the inverse volume to give an overall factor of N, the scattering particle density. The effect of the thermal distribution $\mu(\vec{k})$ of momenta $\vec{k}$ is taken into account by including an integral $\int d\vec{k}\, \mu(\vec{k})$, in accordance with the mixed state procedure of equation (35a). Finally, expressing the S matrix in terms of the scattering amplitude $f(\vec{k}', \vec{k})$, and noting that the squared delta function for energy conservation gives an overall factor of the elapsed time, equation (41d) becomes, in the limit of small elapsed time, a formula for the time derivative of $\rho^{(n)}$. For the $n = 1$ case, the standard answer obtained this way is

$$\frac{\partial \rho^{(1)}(t)_{\vec{R}\vec{R}'}}{\partial t} = -F(\vec{R} - \vec{R}')\rho^{(1)}(t)_{\vec{R}\vec{R}'}, \quad (42a)$$

with

$$F(\vec{R}) = N \int d\vec{k}\, \mu(\vec{k}) \frac{|\vec{k}|}{m} \int d\hat{n}(1 - e^{i(\vec{k} - \hat{n}|\vec{k}|) \cdot \vec{R}}) |f(\hat{n}|\vec{k}|, \vec{k})|^2, \quad (42b)$$

where $\hat{n}$ is a unit vector which gives the direction of the scattered particle momentum $\vec{k}' = \hat{n}|\vec{k}|$. To compare equation (42b) with the $n = 1$ case of equation (41d), we replace $\vec{R}$ by $\vec{R}_2 = \vec{R}_1$ and $\vec{R}'$ by $\vec{R}'_1$. Then we see that the generalization to $n \geqslant 1$ is given by

$$\frac{\partial \rho^{(n)}(t)_{\vec{R}_1 \vec{R}'_1, \ldots, \vec{R}_n \vec{R}'_n}}{\partial t} = -\left[\sum_{\ell=1}^{n} F(\vec{R}_{\ell+1} - \vec{R}'_\ell) \right] \rho^{(n)}(t)_{\vec{R}_1 \vec{R}'_1, \ldots, \vec{R}_n \vec{R}'_n}. \quad (42c)$$

This is our final result for collisional Brownian motion, giving the evolution equation obeyed by the order n density tensor. We see that it has the generic symmetries expected in the quantum noise case: although not totally symmetric in its pair indices, $\rho^{(n)}$ is symmetric under cyclic permutation of these indices. As additional checks, we see that for $n = 2$ the factor involving F is

$$F(\vec{R}_2 - \vec{R}'_1) + F(\vec{R}_1 - \vec{R}'_2), \quad (43a)$$

which is symmetric under the interchange $1 \leftrightarrow 2$, while for $n = 3$ we have

$$F(\vec{R}_2 - \vec{R}'_1) + F(\vec{R}_3 - \vec{R}'_2) + F(\vec{R}_1 - \vec{R}'_3) = F^S + F^A,$$

$$F^S = \frac{1}{2}[F(\vec{R}_2 - \vec{R}'_1) + F(\vec{R}_3 - \vec{R}'_2) + F(\vec{R}_1 - \vec{R}'_3)$$

$$+ F(\vec{R}_1 - \vec{R}'_2) + F(\vec{R}_3 - \vec{R}'_1) + F(\vec{R}_2 - \vec{R}'_3)],$$

$$F^A = \frac{1}{2}[F(\vec{R}_2 - \vec{R}'_1) + F(\vec{R}_3 - \vec{R}'_2) + F(\vec{R}_1 - \vec{R}'_3)$$

$$- F(\vec{R}_1 - \vec{R}'_2) - F(\vec{R}_3 - \vec{R}'_1) - F(\vec{R}_2 - \vec{R}'_3)], \quad (43b)$$

with F^S symmetric, and F^A antisymmetric, under any of the pair interchanges $1 \leftrightarrow 2$ or $1 \leftrightarrow 3$ or $2 \leftrightarrow 3$. Checking the descent equations is easy. Setting $\vec{R}'_1 = \vec{R}_2$, the term $F(\vec{R}_2 - \vec{R}'_1)$ in

equation ($42c$) vanishes, so that on integrating over $\vec{R}'_1$ one is left on the right-hand side with a sum $F(\vec{R}_1 - \vec{R}'_n) + F(\vec{R}_3 - \vec{R}'_2) + \cdots$ that does not involve $\vec{R}'_1$, times

$$\int d\vec{R}'_1 \rho^{(n)}(t)_{\vec{R}_1 \vec{R}'_1, \vec{R}'_1 \vec{R}'_2, \ldots, \vec{R}_n \vec{R}'_n}, \tag{43c}$$

and so the descent equation for $\rho^{(n)}(t)$ then implies the descent equation for its time derivative.

8. The weak coupling Born–Markov approximation and the quantum optical master equation for the density tensor

We turn next to the density tensor extension of the standard weak coupling Born–Markov approximation that is used to give a master equation for the reduced density matrix $\rho^{(1)}$ for a system $\mathcal{S}$ interacting with an environment $\mathcal{E}$. We assume a total system plus environment Hamiltonian $H_{\text{TOT}} = H_{\mathcal{E}} + H_{\mathcal{S}} + H$, with $H_{\mathcal{E}}$ and $H_{\mathcal{S}}$ respectively the environment and system Hamiltonians, and with H the system–environment interaction Hamiltonian. (We omit the customary subscript I on the interaction Hamiltonian to avoid a proliferation of subscripts.) We shall work in this section in interaction picture, in which the operators carry the time dependence associated with $H_{\mathcal{E}}$ and $H_{\mathcal{S}}$. Thus, the interaction Hamiltonian carries a time dependence $H(t)$, and the density matrix obeys the equation of motion

$$\frac{d\rho(t)}{dt} = -i[H(t), \rho(t)] \tag{44a}$$

which can be integrated to give

$$\rho(t) = \rho(0) - i \int_0^t ds [H(s), \rho(s)]. \tag{44b}$$

Substituting equation ($44b$) back into equation ($44a$) gives the additional evolution equation

$$\frac{d\rho(t)}{dt} = -i[H(t), \rho(0)] - \int_0^t ds [H(t), [H(s), \rho(s)]]. \tag{44c}$$

One then notes that up to an error of order H^3 the time argument of the factor $\rho(s)$ in the double commutator term is irrelevant, so this factor can be approximated as $\rho(t)$, giving

$$\frac{d\rho(t)}{dt} = -i[H(t), \rho(0)] - \int_0^t ds [H(t), [H(s), \rho(t)]], \tag{44d}$$

which is used as the starting point for the standard master equation derivation.

Our first step is to derive a suitable extension of equation ($44d$) for the product $\rho_1 \rho_2 \cdots \rho_n$ that appears in equation ($32c$). By the chain rule, we have

$$\frac{d(\rho_1 \rho_2 \cdots \rho_n)}{dt} = \frac{d\rho_1}{dt} \rho_2 \cdots \rho_n + \rho_1 \cdots \frac{d\rho_\ell}{dt} \cdots \rho_n + \rho_1 \cdots \frac{d\rho_n}{dt}. \tag{45a}$$

For each undifferentiated factor on the right of equation ($45a$) we substitute equation ($44b$), and for each time derivative factor we substitute equation ($44c$), with appropriate subscripts added. Let us now organize the terms obtained this way according to the number of factors of H that appear. Since equation ($44c$) contains at least one factor of H, there are no terms in equation ($45a$) with no factors of H. The general term in equation ($45a$) with one factor of H comes from the term in equation ($44c$) with one factor of H, multiplied by the product of the terms from equation ($44b$) with no factors of H, giving

$$-i([H_1(t), \rho_1(0)]\rho_2(0) \cdots \rho_n(0) + \rho_1(0)[H_2(t), \rho_2(0)] \cdots \rho_n(0) + \cdots$$
$$+ \rho_1(0)\rho_2(0) \cdots [H_n(t), \rho_n(0)]). \tag{45b}$$

The terms in equation (45a) with two factors of H are of two types: (1) the quadratic term in H on the right of equation (44c) times factors of $\rho(0)$ and (2) the linear term in H on the right of equation (44b), multiplied by one factor of the linear term on the right of equation (44c), times factors of $\rho(0)$. We now note that up to an error of order H^3, in terms that already contain two factors of H, we can replace all factors $\rho(0)$ or $\rho(s)$ by the corresponding $\rho(t)$, since the differences $\rho(t) - \rho(s)$ and $\rho(t) - \rho(0)$ are all of order H. Collecting everything, we get the following formula, which gives the needed extension of equation (44b),

$$
\begin{aligned}
\frac{d(\rho_1 \rho_2 \cdots \rho_n)}{dt} = {} & -i \sum_{\ell=1}^{n} \rho_1(0) \cdots \rho_{\ell-1}(0)[H_\ell(t), \rho_\ell(0)]\rho_{\ell+1}(0) \cdots \rho_n(0) \\
& - \sum_{\ell=1}^{n} \rho_1(t) \cdots \rho_{\ell-1}(t) \int_0^t ds[H_\ell(t), [H_\ell(s), \rho_\ell(t)]]\rho_{\ell+1}(t) \cdots \rho_n(t) \\
& - \sum_{\ell<m} \Bigg\{ \rho_1(t) \cdots \rho_{\ell-1}(t)[H_\ell(t), \rho_\ell(t)]\rho_{\ell+1}(t) \cdots \rho_{m-1}(t) \\
& \times \int_0^t ds[H_m(s), \rho_m(t)]\rho_{m+1}(t) \cdots \rho_n(t) \\
& + \rho_1(t) \cdots \rho_{\ell-1}(t) \int_0^t ds[H_\ell(s), \rho_\ell(t)]\rho_{\ell+1}(t) \cdots \rho_{m-1}(t) \\
& \times [H_m(t), \rho_m(t)]\rho_{m+1}(t) \cdots \rho_n(t) \Bigg\} + O(H^3).
\end{aligned}
\tag{45c}
$$

Taking the overall $\mathrm{Tr}_\mathcal{E}$ of this expression then gives a formula for the time evolution of $\rho^{(n)}(t)$ as defined by equation (32c).

We now make two standard assumptions. First of all, we assume that at the initial time $t = 0$ the density matrix factorizes so that $\rho(0) = \rho_\mathcal{E} \rho_\mathcal{S}$, with $\rho_\mathcal{E}$ and $\rho_\mathcal{S}$ respectively density matrices for the environment and the system which commute with one another, and with $\rho_\mathcal{E}$ a pure state density matrix obeying $\rho_\mathcal{E}^2 = \rho_\mathcal{E}$. Secondly, we assume that $\langle H \rangle_\mathcal{E} = \mathrm{Tr}_\mathcal{E}\rho_\mathcal{E} H = 0$, that is, we take the interaction Hamiltonian to have a vanishing expectation in the initial environmental state. As a result of these two assumptions, the environmental trace of the first term on the right-hand side of equation (45c) vanishes, since

$$
\begin{aligned}
\mathrm{Tr}_\mathcal{E} \rho_1(0) \cdots & \rho_{\ell-1}(0)[H_\ell(t), \rho_\ell(0)]\rho_{\ell+1}(0) \cdots \rho_n(0) \\
& = \rho_{S1} \cdots \rho_{S\ell-1}[(\mathrm{Tr}_\mathcal{E}\rho_\mathcal{E} H_\ell(t)), \rho_{S\ell}]\rho_{S\ell+1} \cdots \rho_{Sn} = 0.
\end{aligned}
\tag{46a}
$$

The remaining terms in equation (45c) all have two factors of H. Since $\rho(t)$ and $\rho(0)$ differ by one power of H, in these terms, up to an error of order H^3, we can replace all factors $\rho(t)$ by the factorized approximation

$$
\rho(t) \simeq \rho(0) = \rho_\mathcal{E} \rho_\mathcal{S} = \rho_\mathcal{E} \, \mathrm{Tr}_\mathcal{E}\rho(0) \simeq \rho_\mathcal{E} \, \mathrm{Tr}_\mathcal{E}\rho(t) = \rho_\mathcal{E} \rho^{(1)}(t).
\tag{46b}
$$

With these simplifications, and remembering that system operator factors $\rho_\ell^{(1)}$ with different index values ℓ act on different Hilbert spaces $\mathcal{H}_{\mathcal{S};\ell}$ and so commute, equation (45c) becomes an extended version of the Redfield equation,

$$
\begin{aligned}
d\rho^{(n)}(t)/dt = {} & -\sum_{\ell=1}^{n} \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t) \right)_\ell \mathrm{Tr}_\mathcal{E}\rho_\mathcal{E}^{n-1} \int_0^t ds \left[H_\ell(t), \left[H_\ell(s), \rho_\ell^{(1)}(t)\rho_\mathcal{E} \right] \right] \\
& - \sum_{\ell<m} \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t) \right)_{\ell m} \mathrm{Tr}_\mathcal{E} \int_0^t ds \, \rho_\mathcal{E}^{n-(m-\ell)-1} \Big\{ \left[H_\ell(t), \rho_\ell^{(1)}(t)\rho_\mathcal{E} \right] \rho_\mathcal{E}^{m-\ell-1} \\
& \times \left[H_m(s), \rho_m^{(1)}(t)\rho_\mathcal{E} \right] + \left[H_\ell(s), \rho_\ell^{(1)}(t)\rho_\mathcal{E} \right] \rho_\mathcal{E}^{m-\ell-1} \left[H_m(t), \rho_m^{(1)}(t)\rho_\mathcal{E} \right] \Big\}.
\end{aligned}
\tag{46c}
$$

This is converted to the Born–Markov equation by setting $s \to t - s$, and then extending the upper limit of the s integration from t to ∞, giving

$$d\rho^{(n)}(t)/dt = -\sum_{\ell=1}^{n} \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_\ell \mathrm{Tr}_\mathcal{E} \rho_\mathcal{E}^{n-1} \int_0^\infty ds \left[H_\ell(t), \left[H_\ell(t-s), \rho_\ell^{(1)}(t)\rho_\mathcal{E} \right] \right]$$

$$- \sum_{\ell < m} \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_{\ell m} \mathrm{Tr}_\mathcal{E} \int_0^\infty ds\, \rho_\mathcal{E}^{n-(m-\ell)-1}$$

$$\times \left\{ \left[H_\ell(t), \rho_\ell^{(1)}(t)\rho_\mathcal{E} \right] \rho_\mathcal{E}^{m-\ell-1} \left[H_m(t-s), \rho_m^{(1)}(t)\rho_\mathcal{E} \right] \right.$$

$$\left. + \left[H_\ell(t-s), \rho_\ell^{(1)}(t)\rho_\mathcal{E} \right] \rho_\mathcal{E}^{m-\ell-1} \left[H_m(t), \rho_m^{(1)}(t)\rho_\mathcal{E} \right] \right\}. \tag{46d}$$

We now note that equation (46d) can be further simplified, by taking account of the fact that whenever an H factor is sandwiched between factors of $\rho_\mathcal{E}$ it vanishes, since $\rho_\mathcal{E} H \rho_\mathcal{E} = \rho_\mathcal{E} \langle H \rangle_\mathcal{E} = 0$. This eliminates all terms in the sum over ℓ, m that are not adjacent in a cyclic sense, i.e., that do not either have $m = \ell + 1$, $\ell = 1, \dots, n-1$ or $\ell = 1, m = n$. The latter, by use of the cyclic properties of the trace, can be rearranged to give the $\ell = n$ term of the former set. We thus get a simplified set of Born–Markov equations. For $n = 1$, we get the usual starting point for the Born–Markov master equation derivation,

$$d\rho^{(1)}(t)/dt = -\mathrm{Tr}_\mathcal{E} \int_0^\infty ds [H(t)H(t-s)\rho^{(1)}(t)\rho_\mathcal{E} + \rho^{(1)}(t)\rho_\mathcal{E} H(t-s)H(t)$$

$$- H(t)\rho^{(1)}(t)\rho_\mathcal{E} H(t-s) - H(t-s)\rho^{(1)}(t)\rho_\mathcal{E} H(t)], \tag{47a}$$

and for $n \geqslant 2$, with the subscript $n + 1$ identified with 1,

$$d\rho^{(n)}(t)/dt = -\mathrm{Tr}_\mathcal{E}\rho_\mathcal{E} \int_0^\infty ds \sum_{\ell=1}^{n} \left\{ \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_\ell \right.$$

$$\times \left[H_\ell(t)H_\ell(t-s)\rho_\ell^{(1)}(t) + \rho_\ell^{(1)}(t)H_\ell(t-s)H_\ell(t) \right]$$

$$- \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_{\ell\ell+1} \left[\rho_\ell^{(1)}(t)H_\ell(t)H_{\ell+1}(t-s)\rho_{\ell+1}^{(1)}(t) \right.$$

$$\left. \left. + \rho_\ell^{(1)}(t)H_\ell(t-s)H_{\ell+1}(t)\rho_{\ell+1}^{(1)}(t) \right] \right\}. \tag{47b}$$

At this point it is useful to check (and we have done so) that the descent equations are satisfied by equations (47a) and (47b).

The remainder of the derivation follows closely the standard master equation derivation, in the rotating wave approximation, that proceeds from equation (47a), so we will only give a sketch. For further details, and in particular a discussion of the physical justification for the approximations involved, see section 3.3 of [1] and also [13]. One assumes that $H_\ell(t)$ has the form

$$H_\ell(t) = \sum_\alpha \sum_\omega e^{i\omega t} A_{\ell\alpha}^\dagger(\omega) B_\alpha(t), \tag{48a}$$

with $A_{\ell\alpha}^\dagger$ acting only in the system Hilbert space $\mathcal{H}_{\mathcal{S};\ell}$ and with B_α acting only in the environment Hilbert space $\mathcal{H}_\mathcal{E}$, and with the Hermiticity properties $A_{\ell\alpha}^\dagger(\omega) = A_{\ell\alpha}(-\omega)$ and $B_\alpha^\dagger(t) = B_\alpha(t)$. Since equations (47a), (47b) are quadratic in H, one uses equation (48a) twice; for each $H_k(t-s)$ (regardless of the value of the index k) one writes

$$H_k(t-s) = \sum_{\beta\omega} e^{-i\omega(t-s)} A_{k\beta}(\omega) B_\beta(t-s), \tag{48b}$$

and for each $H_k(t)$ (again regardless of the value of k) one writes

$$H_k(t) = \sum_{\alpha\omega'} e^{i\omega' t} A_{k\alpha}^\dagger(\omega') B_\alpha^\dagger(t). \tag{48c}$$

The rotating wave approximation then consists of neglecting terms in the double sum with $\omega' \neq \omega$, so that only the diagonal terms $\omega' = \omega$ are left. From the trace over the environment, and the integral over s, one gets correlators of the form

$$\int_0^\infty ds\, e^{i\omega s} \langle B_\alpha^\dagger(t) B_\beta(t-s) \rangle_{\mathcal{E}} \equiv \Gamma_{\alpha\beta}(\omega),$$

$$\int_0^\infty ds\, e^{i\omega s} \langle B_\beta(t-s) B_\alpha^\dagger(t) \rangle_{\mathcal{E}} = \Gamma_{\alpha\beta}(-\omega)^*, \tag{49a}$$

where in the second line we have used the definition of the first line and the adjointness properties of the integrand. It is also customary to decompose the reservoir correlation function $\Gamma_{\alpha\beta}$ into self-adjoint and anti-self-adjoint parts, according to

$$\Gamma_{\alpha\beta}(\omega) = \tfrac{1}{2}\gamma_{\alpha\beta}(\omega) + iS_{\alpha\beta}(\omega). \tag{49b}$$

Proceeding in this fashion, after some algebra one gets the final result, which can be written as an equation for all $n \geqslant 1$ by including a δ_{n1} to take account of the special nature of the $n = 1$ equation,

$$\begin{aligned}
d\rho^{(n)}(t)/dt = &\sum_{\ell=1}^n \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_\ell i \left[\rho_\ell^{(1)}(t), \sum_{\omega\alpha\beta} S_{\alpha\beta}(\omega) A_{\ell\alpha}^\dagger(\omega) A_{\ell\beta}(\omega) \right] \\
&+ \sum_{\ell=1}^n \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}(\omega) \Bigg\{ \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_\ell \left[\delta_{n1} A_{\ell\beta}(\omega) \rho_\ell^{(1)}(t) A_{\ell\alpha}^\dagger(\omega) \right. \\
&\left. - \frac{1}{2} \{ A_{\ell\alpha}^\dagger(\omega) A_{\ell\beta}(\omega), \rho_\ell^{(1)}(t) \} \right] \\
&+ \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_{\ell\,\ell+1} \rho_\ell^{(1)}(t) A_{\ell\alpha}^\dagger(\omega) A_{\ell+1\beta}(\omega) \rho_{\ell+1}^{(1)}(t) \Bigg\}. \tag{50a}
\end{aligned}$$

Despite the fact that the $n = 1$ and $n \geqslant 2$ density tensors have a different structure, the descent equations are satisfied by equation (50a), as verified in appendix D.

Finally, we note that equation (50a) is readily converted to the quantum optical master equation and its density tensor generalizations, by taking α to be a three-vector index, so that A_α becomes $\vec{A}$, which is related to the dipole operator by equation (3.182) of [1]. Also, one takes $S_{\alpha\beta}(\omega) = \delta_{\alpha\beta} S(\omega)$, with $S(\omega)$ given by equation (3.205) of [1], and $\gamma_{\alpha\beta}(\omega) = (4\omega^3/3)[1 + N(\omega)]\delta_{\alpha\beta}$, with $N(\omega) = 1/(e^{\beta\omega} - 1)$ the photon number operator. One gets in this way the density tensor generalization of the quantum optical master equation,

$$\begin{aligned}
d\rho^{(n)}(t)/dt = &\sum_{\ell=1}^n \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_\ell i \left[\rho_\ell^{(1)}(t), \sum_\omega S(\omega) \vec{A}_\ell^\dagger(\omega) \cdot \vec{A}_\ell(\omega) \right] \\
&+ \sum_{\ell=1}^n \sum_\omega (4\omega^3/3)[1 + N(\omega)] \Bigg\{ \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_\ell \left[\delta_{n1} \vec{A}_\ell(\omega) \cdot \rho_\ell^{(1)}(t) \vec{A}_\ell^\dagger(\omega) \right. \\
&\left. - \frac{1}{2} \{ \vec{A}_\ell^\dagger(\omega) \cdot \vec{A}_\ell(\omega), \rho_\ell^{(1)}(t) \} \right] \\
&+ \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_{\ell\,\ell+1} \rho_\ell^{(1)}(t) \vec{A}_\ell^\dagger(\omega) \cdot \vec{A}_{\ell+1}(\omega) \rho_{\ell+1}^{(1)}(t) \Bigg\}, \tag{50b}
\end{aligned}$$

which is our final result of this section.

9. The Caldeira–Leggett model master equation for the density tensor

The Caldeira–Leggett model [14] describes the damping of the one-dimensional motion of a Brownian particle of mass m, moving in a potential $V(x)$ and interacting with an environment consisting of harmonic oscillators with masses m_o and frequencies ω_o, and annihilation operator b_o. The interaction Hamiltonian is assumed to be a linear coupling $H = -xB$, with

$$B = \sum_o \kappa_o x_o = \sum_o \kappa_o \big(b_o + b_o^\dagger\big)\big/(2m_o\omega_o)^{\frac{1}{2}} \tag{51a}$$

a weighted sum of the harmonic oscillator coordinates. A counter-term formally of order H^2,

$$H_c = x^2 \sum_o \frac{\kappa_o^2}{2m_o\omega_o} \equiv x^2 C, \tag{51b}$$

is included in the calculation, so that the total Hamiltonian is

$$H_{\text{TOT}} = H_{\mathcal{E}} + H_{\mathcal{S}} + H + H_c, \tag{52a}$$

with $H_{\mathcal{E}}$ and $H_{\mathcal{S}}$ respectively the oscillator and particle Hamiltonians,

$$H_{\mathcal{E}} = \sum_o \omega_o \left(b_o^\dagger b_o + \frac{1}{2}\right),$$
$$H_{\mathcal{S}} = \frac{p^2}{2m} + V(x). \tag{52b}$$

Our aim will be to get a description of the effect on the particle motion of the couplings to the oscillator environment, in the high temperature limit. Our derivation of the density tensor generalization of the high temperature master equation closely follows that of section 3.6 of [1], to which the reader is referred for a discussion of the physical motivation of the approximations involved.

Since the environmental expectation of the interaction Hamiltonian H vanishes, we can proceed directly from the simplified Born–Markov equation of equations (47a) and (47b). The first step is to transform the density matrix $\rho(t)$ back to Schrödinger picture; it is easy to see that the effect of this is to replace $H(t)$ by $H = H(0)$, to replace $H(t-s)$ by $H(-s)$ (with $H(-s)$ still in the interaction picture) and to change $\mathrm{d}/\mathrm{d}t$ to $\mathrm{D}/\mathrm{d}t$, defined by

$$\mathrm{D}\rho^{(n)}(t)/\mathrm{d}t = \mathrm{d}\rho^{(n)}(t)/\mathrm{d}t + \mathrm{i}\sum_{\ell=1}^{n} \mathrm{Tr}_{\mathcal{E}}\rho_1(t)\cdots\rho_{\ell-1}(t)$$
$$\times \big[p_\ell^2\big/(2m) + V(x_\ell), \rho_\ell(t)\big]\rho_{\ell+1}(t)\cdots\rho_n(t). \tag{53a}$$

It is also necessary to explicitly include commutators arising from the counter-term, which is easy since this term is treated as being already quadratic in H. For the analogue of equation (47a) for the special case $n = 1$, we find

$$\mathrm{D}\rho^{(1)}(t)/\mathrm{d}t = -\mathrm{i}[H_c, \rho^{(1)}(t)] - \mathrm{Tr}_{\mathcal{E}}\int_0^\infty \mathrm{d}s[HH(-s)\rho^{(1)}(t)\rho_{\mathcal{E}} + \rho^{(1)}(t)\rho_{\mathcal{E}}H(-s)H$$
$$- H\rho^{(1)}(t)\rho_{\mathcal{E}}H(-s) - H(-s)\rho^{(1)}(t)\rho_{\mathcal{E}}H], \tag{53b}$$

and for the analogue of equation (47b) for $n \geqslant 2$, we have

$$\mathrm{d}\rho^{(n)}(t)/\mathrm{d}t = -\mathrm{i}\sum_{\ell=1}^{n} \big(\rho_1^{(1)}(t)\cdots\rho_n^{(1)}(t)\big)_\ell\big[H_{c\ell}, \rho_\ell^{(1)}(t)\big] - \mathrm{Tr}_{\mathcal{E}}\rho_{\mathcal{E}}\int_0^\infty \mathrm{d}s \sum_{\ell=1}^{n}$$
$$\times \big\{\big(\rho_1^{(1)}(t)\cdots\rho_n^{(1)}(t)\big)_\ell\big[H_\ell H_\ell(-s)\rho_\ell^{(1)}(t) + \rho_\ell^{(1)}(t)H_\ell(-s)H_\ell\big]$$

$$- \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_{\ell\ell+1} \left[\rho_\ell^{(1)}(t) H_\ell H_{\ell+1}(-s)\rho_{\ell+1}^{(1)}(t)\right.$$

$$+ \left.\rho_\ell^{(1)}(t) H_\ell(-s) H_{\ell+1}\rho_{\ell+1}^{(1)}(t)\right]\}. \tag{53c}$$

We next note that

$$H_\ell = -x_\ell(0)B(0), \qquad H_\ell(-s) = -x_\ell(-s)B(-s), \tag{54a}$$

where, using the assumption that the system evolution is slow compared to the oscillator time scale, we approximate $x_\ell(-s)$ by its free particle dynamics,

$$x_\ell(-s) \simeq x_\ell - \frac{p_\ell}{m}s. \tag{54b}$$

Since the right-hand sides of equations (53b), (53c) are quadratic in H, the operator B giving the coupling to the oscillators appears, after the environmental trace is taken, only through the correlators

$$D(s) \equiv i\langle[B(0), B(-s)]\rangle_\mathcal{E},$$

$$D_1(s) \equiv \langle\{B(0), B(-s)\}\rangle_\mathcal{E}, \tag{55a}$$

so that we have

$$\langle B(0)B(-s)\rangle_\mathcal{E} = \tfrac{1}{2}[D_1(s) - iD(s)],$$

$$\langle B(-s)B(0)\rangle_\mathcal{E} = \tfrac{1}{2}[D_1(s) + iD(s)]. \tag{55b}$$

These correlators appear in the following integrals, which are evaluated or approximated in section 3.6.2 of [1],

$$\int_0^\infty ds\, D(s) = 2C,$$

$$\int_0^\infty ds\, D_1(s) = 4m\gamma k_B T,$$

$$\int_0^\infty ds\, s D(s) = 2m\gamma,$$

$$\int_0^\infty ds\, s D_1(s) = 4m\gamma k_B T/\Omega \simeq 0, \tag{55c}$$

with C the constant defined by the counter-term of equation (51b), with γ a constant determined by the harmonic oscillator spectral density, with k_B and T respectively the Boltzmann constant and environment temperature and with Ω a frequency cutoff. For a spectral density $J(\omega)$ with a Lorentz–Drude cutoff function, one has

$$J(\omega) = \sum_o \frac{\kappa_o^2}{2m_o\omega_o}\delta(\omega - \omega_o) = \frac{2m\gamma}{\pi}\omega\frac{\Omega^2}{\Omega^2 + \omega^2}. \tag{55d}$$

This completes the specification of the calculation; the rest is just the algebra of assembling all the pieces, and so we pass directly to the result. For $n = 1$, we get the Caldeira–Leggett master equation,

$$D\rho^{(1)}(t)/dt = -i\gamma[x, \{p, \rho^{(1)}(t)\}] - 2m\gamma k_B T[x, [x, \rho^{(1)}(t)]]. \tag{56a}$$

For the density tensors with $n \geqslant 2$, we correspondingly get

$$D\rho^{(n)}/dt = \sum_{\ell=1}^n \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_\ell \left[-2m\gamma k_B T\left\{x_\ell^2, \rho_\ell^{(1)}(t)\right\}\right.$$

$$+ i\gamma\left.\left(\rho_\ell^{(1)}(t)p_\ell x_\ell - x_\ell p_\ell \rho_\ell^{(1)}(t)\right)\right]$$

$$+ \sum_{\ell=1}^n \left(\rho_1^{(1)}(t) \cdots \rho_n^{(1)}(t)\right)_{\ell\ell+1} \left[4m\gamma k_B T\rho_\ell^{(1)}(t)x_\ell x_{\ell+1}\rho_{\ell+1}^{(1)}(t)\right.$$

$$+ i\gamma\rho_\ell^{(1)}(t)(x_\ell p_{\ell+1} - p_\ell x_{\ell+1})\rho_{\ell+1}^{(1)}(t)\bigg]. \tag{56b}$$

We also note that the term proportional to γ on the first line of equation (56b) can be written in the alternative form

$$i\gamma\left(\rho_\ell^{(1)}(t)p_\ell x_\ell - x_\ell p_\ell \rho_\ell^{(1)}(t)\right) = \gamma\rho_\ell^{(1)}(t) + \frac{i}{2}\gamma\left[\rho_\ell^{(1)}(t), \{x_\ell, p_\ell\}\right]. \tag{56c}$$

Equations (56a) and (56b) are our final results for the Caldeira–Leggett model. As was the case for the master equations derived in the preceding section, despite the differences between the structure of the $n = 1$ and the $n \geqslant 2$ equations, the descent equations are satisfied, as verified in appendix E.

10. An application to state vector reduction

We turn now to considerations that bridge the discussions given above in the classical and quantum noise cases. We begin with an analysis of two Itô stochastic Schrödinger equations,

$$d|\psi\rangle = -\tfrac{1}{2}(A - \langle A\rangle)^2\,dt|\psi\rangle + (A - \langle A\rangle)\,dW_t|\psi\rangle, \tag{57a}$$

and

$$d|\psi\rangle = -\tfrac{1}{2}A^2\,dt|\psi\rangle + iA\,dW_t|\psi\rangle, \tag{57b}$$

with dW_t a real Brownian noise obeying $dW_t^2 = dt$, and where we have dropped the Hamiltonian term. These lead to the respective density matrix evolution equations

$$d\rho = -\tfrac{1}{2}[A, [A, \rho]]\,dt + [\rho, [\rho, A]]\,dW_t, \tag{58a}$$

and

$$d\rho = -\tfrac{1}{2}[A, [A, \rho]]\,dt + i[A, \rho]\,dW_t, \tag{58b}$$

which correspond to the same Lindblad-type evolution equation for the expectation $E[\rho]$,

$$dE[\rho] = \mathcal{L}E[\rho]\,dt, \qquad \mathcal{L}\rho = -\tfrac{1}{2}[A, [A, \rho]]. \tag{58c}$$

Let us now consider the effect of the stochastic evolutions of equations (58a), (58b), (58c) on the expectation of the variance $V = \mathrm{Var}(A)$ of the operator A,

$$V = \mathrm{Tr}\,\rho A^2 - (\mathrm{Tr}\,\rho A)^2,$$
$$E[V] = \mathrm{Tr}\,E[\rho]A^2 - E\left[\rho_{i_1 j_1}\rho_{i_2 j_2}\right]A_{j_1 i_1}A_{j_2 i_2}$$
$$= \mathrm{Tr}\,\rho^{(1)}A^2 - \rho^{(2)}_{i_1 j_1, i_2 j_2}A_{j_1 i_1}A_{j_2 i_2}, \tag{59a}$$

where in the final line we have used the density tensor definition of equation (15). For the time evolution of $E[V]$ we have

$$dE[V]/dt = \mathrm{Tr}(\mathcal{L}E[\rho])A^2 - d\rho^{(2)}_{i_1 j_1, i_2 j_2}A_{j_1 i_1}A_{j_2 i_2}$$
$$= \mathrm{Tr}(\mathcal{L}E[\rho])A^2 - 2E\left[\rho_{i_1 j_1}(\mathcal{L}\rho)_{i_2 j_2}\right]A_{j_1 i_1}A_{j_2 i_2} - E\left[C_{i_1 j_1, i_2 j_2}\right]A_{j_1 i_1}A_{j_2 i_2}, \tag{59b}$$

where we have used equation (58c) in the first line and equation (23a) in the second line. Since the cyclic property of the trace implies that $\mathrm{Tr}[A, [A, \rho]]A = 0$, the terms in equation (59b) involving the Lindblad $\mathcal{L}$ all vanish, and so the time derivative of $E[V]$ comes entirely form the final term,

$$dE[V]/dt = -E\left[C_{i_1 j_1, i_2 j_2}\right]A_{j_1 i_1}A_{j_2 i_2}, \tag{59c}$$

and thus is determined by the evolution equation for the second-order density tensor. This is why the state vector evolutions of equations (57a) and (57b), or equivalently the density matrix evolutions of equations (58a) and (58b), lead to very different results for the evolution of the variance of the operator A. The tensor $C_{i_1 j_1, i_2 j_2}$ corresponding to equations (57b) and (58b) is

given in equation (24b), and since the cyclic property of the trace implies that $\mathrm{Tr}[A, \rho]A = 0$, one has $\mathrm{d}E[V]/\mathrm{d}t = 0$ for this evolution. On the other hand, the tensor $C_{i_1 j_1, i_2 j_2}$ corresponding to equations (57a) and (58a) is given in equation (24a), and through equation (59c) implies that

$$\mathrm{d}E[V]/\mathrm{d}t = -E[(\mathrm{Tr}[\rho, [\rho, A]]A)^2] = -E[(\mathrm{Tr}([\rho, A])^2)^2], \tag{59d}$$

which is negative definite. Starting from equation (59d), some simple inequalities imply that the stochastic evolution of equations (57a) and (58a) drives the variance of A to zero as $t \to \infty$, and hence reduces the state vector to an eigenstate of A, as discussed in detail in [15].

Let us now consider a quantum system $\mathcal{S}$, consisting of a microscopic system coupled to a macroscopic measuring apparatus, interacting with a quantum environment $\mathcal{E}$, with the totality forming a closed system. A general result [16], using just the linearity of quantum mechanics, shows that state vector reduction cannot occur in this case. To understand this result through an analysis similar to that just given for equations (57a), (57b), let us consider the behaviour of the variance of a system operator A which is a good 'pointer observable'. By definition, a system operator commutes with the environment Hamiltonian $H_{\mathcal{E}}$, and since the system in this case includes the apparatus and so is macroscopic, the pointer observable also obeys [17] $[A, H] = 0$, with H the system–environment interaction Hamiltonian. Let us now write the density matrix evolution in Schrödinger picture,

$$\mathrm{d}\rho/\mathrm{d}t = -\mathrm{i}[H_{\mathrm{TOT}}, \rho] = -\mathrm{i}[H_{\mathcal{S}} + H_{\mathcal{E}} + H, \rho], \tag{60a}$$

with $H_{\mathcal{S}}$ the system Hamiltonian. We consider the system evolution after a brief interaction has entangled the apparatus states with the microscopic subsystem quantum states that are to be distinguished by the pointer reading. For the time evolution of the variance of the pointer observable A, we have

$$\mathrm{d}V/\mathrm{d}t = \mathrm{Tr}(\mathrm{d}\rho/\mathrm{d}t)A^2 - 2(\mathrm{Tr}\,\rho A)(\mathrm{Tr}(\mathrm{d}\rho/\mathrm{d}t)A), \tag{60b}$$

which substituting equation (60a), and using the cyclic property of the trace and the fact that A commutes with both $H_{\mathcal{E}}$ and H, simplifies to

$$\mathrm{d}V/\mathrm{d}t = \mathrm{i}\,\mathrm{Tr}\,\rho[H_{\mathcal{S}}, A^2] - 2\mathrm{i}(\mathrm{Tr}\,\rho A)(\mathrm{Tr}\,\rho[H_{\mathcal{S}}, A]). \tag{60c}$$

This can be further simplified by using the definition of the reduced density matrix $\rho^{(1)} = \mathrm{Tr}_{\mathcal{E}}\rho$, together with the fact that the commutators in equation (60c) involve only system operators, giving

$$\mathrm{d}V/\mathrm{d}t = \mathrm{i}\,\mathrm{Tr}_{\mathcal{S}}\rho^{(1)}[H_{\mathcal{S}}, A^2] - 2\mathrm{i}(\mathrm{Tr}_{\mathcal{S}}\rho^{(1)}A)(\mathrm{Tr}_{\mathcal{S}}\rho^{(1)}[H_{\mathcal{S}}, A]). \tag{60d}$$

We see that, unlike the Itô equation case discussed above, the time derivative of V here is determined by $\rho^{(1)}$ rather than by $\rho^{(2)}$.

Let us now take the pointer observable to be a pointer centre of mass coordinate $A = X$, in which case, once the entanglement of the pointer with the microsystem being measured has been established, the relevant part of the system Hamiltonian $H_{\mathcal{S}}$ is $P^2/(2M)$, with P the total momentum operator for the pointer of macroscopic mass M. Evaluating the commutators, and writing $\mathrm{Tr}_{\mathcal{S}}\rho^{(1)}\mathcal{O} = \langle \mathcal{O} \rangle$, we see that

$$\begin{aligned} \mathrm{d}V/\mathrm{d}t &= (1/M)\langle \{P, X\} \rangle - (2/M)\langle X \rangle \langle P \rangle \\ &= (1/M)\langle \{X - \langle X \rangle, P - \langle P \rangle\} \rangle. \end{aligned} \tag{61a}$$

By the Schwartz inequality, the right-hand side of equation (61a) is bounded by

$$(2/M)\langle (X - \langle X \rangle)^2 \rangle^{\frac{1}{2}}\langle (P - \langle P \rangle)^2 \rangle^{\frac{1}{2}} = (2/M)\Delta X \Delta P. \tag{61b}$$

Let us now determine the minimum value of the bound of equation (61b) that is compatible with the parameters of a feasible measurement. Since the uncertainty principle implies that

$\Delta X \Delta P \geqslant 1/2$, we get a least upper bound on equation (61*b*) by substituting $\Delta X \Delta P \sim 1/2$. This shows that $|dV/dt|$ can be made as small as $\sim 1/M$, which since M is macroscopic can be made essentially arbitrarily small[3]. Hence, the variance of the pointer variable A stays essentially constant and is not forced to reduce to zero in the course of the measurement.

We conclude, in agreement with the arguments of [16], that a quantum apparatus interacting with a quantum environment does not act like the stochastic equation of equation (57*a*) in terms of reducing the state vector. Although a quantum environment acts on a quantum system with a form of 'noise', our analysis of the density tensor hierarchy in the classical and quantum noise cases shows that structures with different kinematical symmetries[4], different dynamical evolutions and different implications for the measurement process are involved. As a result, the quantum noise in a closed quantum system does not mimic the action of the classical noise in objective reduction models and cannot be invoked to give a resolution of the quantum measurement problem within the framework of unmodified quantum mechanics.

Acknowledgments

Appendix A. Descent equations for the isotropic spin-1/2 ensemble

Let us write the generating function of equation (14*b*) as

$$G[a_{ij}] = fg, \qquad f(x) = \sinh x^{\frac{1}{2}}/x^{\frac{1}{2}},$$
$$x = \vec{A}^2, \qquad g = e^{\frac{1}{2}\,\mathrm{Tr}a}. \tag{A.1}$$

Then, we find

$$\frac{\partial G}{\partial a_{mr}} = \frac{1}{2}\delta_{mr}G + gf'\vec{A}\cdot\vec{\sigma}_{mr},$$
$$\frac{\partial^2 G}{\partial a_{mr}\partial a_{pq}} = \frac{1}{2}\delta_{mr}\frac{\partial G}{\partial a_{pq}} + \frac{1}{2}\delta_{pq}gf'\vec{A}\cdot\vec{\sigma}_{mr} + gf''\vec{A}\cdot\vec{\sigma}_{mr}\vec{A}\cdot\vec{\sigma}_{pq} + \frac{1}{2}gf'\vec{\sigma}_{mr}\cdot\vec{\sigma}_{pq}. \tag{A.2}$$

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[3] Restoring factors of Planck's constant, $|dV/dt|$ can be as small as $\hbar/M$, for which the reduction time dt is at least of order $M\,dV/\hbar$. For $M \sim 10^{24}m_{\mathrm{proton}}$ and $dV \sim (1\ \mathrm{cm})^2$, this gives $dt \sim M(1\ \mathrm{cm})^2/\hbar \sim 10^{27}$ s $\sim 10^{10}$ times the age of the universe. Note that our argument places no restriction on the mean pointer momentum $\langle P\rangle$ that establishes the time needed to attain one or the other of the measurement outcomes X starting from the initial pointer position.

[4] The dissimilarities between the symmetries of the classical noise and quantum noise hierarchies are least for the order two density tensor. In the order two case, cyclic symmetry is equivalent to full permutation symmetry, and so the index symmetry properties are the same in the classical and quantum noise cases, and as a consequence the descent equations in the quantum noise case correspond to the idempotence descent equations in the classical noise case. Only the classical noise descent equation implied by the unit trace condition has no precise quantum noise counterpart: in the classical case, one has

$$\delta_{i_1 j_1}\rho^{(2)}_{i_1 j_1, i_2 j_2} = \delta_{i_1 j_1}E[\rho_{i_1 j_1}\rho_{i_2 j_2}] = E[\rho_{i_2 j_2}] = \rho^{(1)}_{i_2 j_2}$$

whereas in the quantum noise case one instead has

$$\delta_{i_1 j_1}\rho^{(2)}_{i_1 j_1, i_2 j_2} = \delta_{i_1 j_1}\mathrm{Tr}_{\mathcal{E}}\,\rho_{i_1 j_1}\rho_{i_2 j_2} = \mathrm{Tr}_{\mathcal{E}}\,\rho_{\mathcal{E}}\rho_{i_2 j_2} \neq \mathrm{Tr}_{\mathcal{E}}\,\rho_{i_2 j_2} = \rho^{(1)}_{i_2 j_2},$$

with $\rho_{\mathcal{E}} = \mathrm{Tr}_{\mathcal{S}}\rho$ the reduced density matrix of the environment with the system traced out.

Here, the primes denote derivatives of f with respect to x, and in this notation f obeys the second-order differential equation

$$xf'' + \tfrac{3}{2}f' = \tfrac{1}{4}f. \tag{A.3}$$

Contracting the first expression in equation (A.2) with δ_{mr}, and using the tracelessness of the Pauli matrices, gives the first equation in equation (7b). Contracting the second expression in equation (A.2) with δ_{rp}, and using the differential equation of equation (A.3) together with the Pauli matrix identities $(\vec{\sigma}^2)_{mq} = 3\delta_{mq}$ and $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, which implies $\vec{A} \cdot \vec{\sigma}_{mp} \vec{A} \cdot \vec{\sigma}_{pq} = \vec{A}^2 \delta_{mq}$, gives the second equation in equation (7b).

Appendix B. Descent equations for the Itô stochastic Schrödinger equation

We wish here to verify that

$$dG[a] = dt\, E\left[\left(a_{mr}(\mathcal{L}\rho)_{mr} + \tfrac{1}{2}a_{mr}a_{pq}C_{mr,pq}\right)e^{\rho \cdot a}\right] \tag{B.1}$$

obeys the descent equations of equation (7b). Since

$$\delta_{mr}(\mathcal{L}\rho)_{mr} = \delta_{mr}C_{mr,pq} = \delta_{pq}C_{mr,pq} = 0, \tag{B.2a}$$

we have

$$\delta_{uv}\frac{\partial\, dG[a]}{\partial a_{uv}} = dt\, E\left[\left(a_{mr}(\mathcal{L}\rho)_{mr} + \frac{1}{2}a_{mr}a_{pq}C_{mr,pq}\right)(\text{Tr}\,\rho)\, e^{\rho \cdot a}\right] = dG[a], \tag{B.2b}$$

giving the first identity in equation (7b). Next we calculate

$$\frac{\partial\, dG[a]}{\partial a_{mq}} = dt\, E\left[\left((\mathcal{L}\rho)_{mq} + \frac{1}{2}a_{uv}(C_{mq,uv} + C_{uv,mq})\right)e^{\rho \cdot a}\right.$$
$$\left. + \left(a_{uv}(\mathcal{L}\rho)_{uv} + \frac{1}{2}a_{uv}a_{rs}C_{uv,rs}\right)\rho_{mq}\, e^{\rho \cdot a}\right], \tag{B.3a}$$

while for the contraction of the second variation we have (with indices m, q implicit on the right-hand side)

$$\frac{\partial^2\, dG[a]}{\partial a_{mr}\partial a_{rq}} = dt\, E[(S_1 + S_2 + a_{uv}(T_{1uv} + T_{2uv}))\, e^{\rho \cdot a}], \tag{B.3b}$$

with

$$\begin{aligned}
S_1 &= \tfrac{1}{2}(C_{mr,rq} + C_{rq,mr}),\\
S_2 &= \{\mathcal{L}\rho, \rho\}_{mq},\\
T_{1uv} &= \tfrac{1}{2}[(C_{mr,uv} + C_{uv,mr})\rho_{rq} + \rho_{mr}(C_{uv,rq} + C_{rq,uv})],\\
T_{2uv} &= \left[(\mathcal{L}\rho)_{uv} + \tfrac{1}{2}a_{rs}C_{uv,rs}\right]\rho_{mq}.
\end{aligned} \tag{B.3c}$$

We see immediately that $a_{uv}T_{2uv}$ gives all of the second line of equation (B.3a). From equations (16b) and (18) we find

$$\{\mathcal{L}\rho, \rho\}_{mq} = (\mathcal{L}\rho)_{mq} - [(c_k - \langle c_k\rangle)\rho(c_k - \langle c_k\rangle)^\dagger]_{mq} - \rho_{mq}\langle(c_k - \langle c_k\rangle)^\dagger(c_k - \langle c_k\rangle)\rangle, \tag{B.4a}$$

while from equation (21b) we have

$$\tfrac{1}{2}(C_{mr,rq} + C_{rq,mr}) = [(c_k - \langle c_k\rangle)\rho(c_k - \langle c_k\rangle)^\dagger]_{mq} + \rho_{mq}\langle(c_k - \langle c_k\rangle)^\dagger(c_k - \langle c_k\rangle)\rangle. \tag{B.4b}$$

Hence $S_1 + S_2 = (\mathcal{L}\rho)_{mq}$, giving the $\mathcal{L}\rho$ part of the first line of equation (B.3a). Finally, again using equation (21b) we find that

$$\tfrac{1}{2}[(C_{mr,uv} + C_{uv,mr})\rho_{rq} + \rho_{mr}(C_{uv,rq} + C_{rq,uv})] = \tfrac{1}{2}(C_{mq,uv} + C_{uv,mq}), \qquad (\text{B.}4c)$$

and so $a_{uv}T_{1uv}$ gives the remainder of the first line of equation (B.3a), completing the check of the descent equations.

Appendix C. Descent equations for the jump Schrödinger equation

We verify here that

$$dG[a] = dt\, E\left[\left(a \cdot \mathcal{L}\rho + \sum_{p=2}^{\infty}\sum_{k} v_k \frac{(a \cdot Q_k)^p}{p!}\right) e^{a \cdot \rho}\right] \qquad (\text{C.}1a)$$

obeys the descent equations of equation (7b). Since $\mathrm{Tr}(\mathcal{L}\rho) = 0$ and $\mathrm{Tr}\, Q_k = \langle B_k + B_k^{\dagger} + B_k^{\dagger}B_k\rangle = 0$, we have

$$\delta_{uv}\frac{\partial dG[a]}{\partial a_{uv}} = dt\, E\left[\left(a \cdot \mathcal{L}\rho + \sum_{p=2}^{\infty}\sum_{k} v_k \frac{(a \cdot Q_k)^p}{p!}\right)(\mathrm{Tr}\,\rho)\, e^{a \cdot \rho}\right], \qquad (\text{C.}1b)$$

checking the first line of equation (7b). Next we calculate the first variation of G,

$$\frac{\partial dG[a]}{\partial a_{mq}} = dt\, E\left[\left((\mathcal{L}\rho)_{mq} + \sum_{p=2}^{\infty}\sum_{k} v_k \frac{(a \cdot Q_k)^{p-1}}{(p-1)!}(Q_k)_{mq}\right) e^{a \cdot \rho}\right.$$
$$\left. + \left(a \cdot \mathcal{L}\rho + \sum_{p=2}^{\infty}\sum_{k} v_k \frac{(a \cdot Q_k)^p}{p!}\right)\rho_{mq}\, e^{a \cdot \rho}\right], \qquad (\text{C.}2a)$$

and the contracted second variation,

$$\frac{\partial^2\, dG[a]}{\partial a_{mr}\partial a_{rq}} = dt\, E[(S_1 + S_2 + S_3 + S_4)\, e^{a \cdot \rho}], \qquad (\text{C.}2b)$$

with

$$S_1 = \sum_{p=2}^{\infty}\sum_{k} v_k \frac{(a \cdot Q_k)^{p-2}}{(p-2)!}(Q_k^2)_{mq},$$

$$S_2 = \{\mathcal{L}\rho, \rho\}_{mq},$$

$$S_3 = \sum_{p=2}^{\infty}\sum_{k} v_k \frac{(a \cdot Q_k)^{p-1}}{(p-1)!}\{Q_k, \rho\}_{mq}, \qquad\qquad (\text{C.}2c)$$

$$S_4 = \left(a \cdot \mathcal{L}\rho + \sum_{p=2}^{\infty}\sum_{k} v_k \frac{(a \cdot Q_k)^p}{p!}\right)\rho_{mq}$$

We see immediately that S_4 gives all of the second line of equation (C.2a). From equation (30c), which we rewrite here,

$$\{\mathcal{L}\rho, \rho\} = \mathcal{L}\rho - \sum_{k} v_k Q_k^2,$$
$$\{Q_k, \rho\} = Q_k - Q_k^2, \qquad\qquad (\text{C.}3a)$$

we see that the $\mathcal{L}\rho$ part of S_2 and the Q_k part of $\{Q_k, \rho\}$ in S_3 give the first line of equation (C.2a). To complete the verification, we must show that S_1 cancels against the remainder of $S_2 + S_3$, which is

$$-\sum_k v_k Q_k^2 - \sum_{p=2}^{\infty}\sum_k v_k \frac{(a \cdot Q_k)^{p-1}}{(p-1)!} (Q_k^2)_{mq}. \tag{C.3b}$$

But separating off the $p = 2$ term of S_1, and making the change of variable $p \to p + 1$ in the remaining sum, we see that S_1 is exactly the negative of equation (C.3b), completing the argument.

Appendix D. Descent equations for the Born–Markov master equation

We wish here to verify that equation (50a) obeys the descent equations of equation (34). We separate the verification into two parts, first checking the descent from $n = 2$ to $n = 1$, and then checking the descent from general $n > 2$ to $n - 1$. For the $n = 2$ density tensor time derivative, writing out all terms in equation (50a) explicitly, and using the fact that since operators labelled with subscripts 2 and 1 act on different Hilbert spaces, the order in which they are written is irrelevant, we have

$$\begin{aligned}
\mathrm{d}\rho^{(2)}(t)/\mathrm{d}t = {}& \mathrm{i}\left[\rho_1^{(1)}(t), \sum_{\omega\alpha\beta} S_{\alpha\beta}(\omega) A_{1\alpha}^{\dagger}(\omega) A_{1\beta}(\omega)\right]\rho_2^{(1)}(t) \\
& + \rho_1^{(1)}(t)\mathrm{i}\left[\rho_2^{(1)}(t), \sum_{\omega\alpha\beta} S_{\alpha\beta}(\omega) A_{2\alpha}^{\dagger}(\omega) A_{2\beta}(\omega)\right] \\
& - \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}(\omega)\frac{1}{2}\{A_{1\alpha}^{\dagger}(\omega) A_{1\beta}(\omega), \rho_1^{(1)}(t)\}\rho_2^{(1)}(t) \\
& - \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}(\omega)\rho_1^{(1)}(t)\frac{1}{2}\{A_{2\alpha}^{\dagger}(\omega) A_{2\beta}(\omega), \rho_2^{(1)}(t)\} \\
& + \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}(\omega)\left[\rho_1^{(1)}(t) A_{1\alpha}^{\dagger}(\omega) A_{2\beta}(\omega)\rho_2^{(1)}(t) + A_{1\beta}(\omega)\rho_1^{(1)}(t)\rho_2^{(1)}(t) A_{2\alpha}^{\dagger}(\omega)\right].
\end{aligned} \tag{D.1a}$$

Contracting the column index associated with the subscript 1 with the row index associated with the subscript 2, and dropping the subscripts since all operators now act in the same Hilbert space, we get

$$\begin{aligned}
\mathrm{d}\rho^{(2)}(t)/\mathrm{d}t \to {}& \mathrm{i}\left[\rho^{(1)}(t), \sum_{\omega\alpha\beta} S_{\alpha\beta}(\omega) A_{\alpha}^{\dagger}(\omega) A_{\beta}(\omega)\right]\rho^{(1)}(t) \\
& + \rho^{(1)}(t)\mathrm{i}\left[\rho^{(1)}(t), \sum_{\omega\alpha\beta} S_{\alpha\beta}(\omega) A_{\alpha}^{\dagger}(\omega) A_{\beta}(\omega)\right] \\
& - \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}(\omega)\frac{1}{2}\{A_{\alpha}^{\dagger}(\omega) A_{\beta}(\omega), \rho^{(1)}(t)\}\rho^{(1)}(t) \\
& - \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}(\omega)\rho^{(1)}(t)\frac{1}{2}\{A_{\alpha}^{\dagger}(\omega) A_{\beta}(\omega), \rho^{(1)}(t)\}
\end{aligned}$$

$$+ \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}(\omega)\left[\rho^{(1)}(t)A_\alpha^\dagger(\omega)A_\beta(\omega)\rho^{(1)}(t) + A_\beta(\omega)(\rho^{(1)}(t))^2 A_\alpha^\dagger(\omega)\right]$$

$$= \mathrm{i}\left[(\rho^{(1)}(t))^2, \sum_{\omega\alpha\beta} S_{\alpha\beta}(\omega)A_\alpha^\dagger(\omega)A_\beta(\omega)\right]$$

$$+ \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}(\omega)\left[A_\beta(\omega)(\rho^{(1)}(t))^2 A_\alpha^\dagger(\omega) - \frac{1}{2}\{(\rho^{(1)}(t))^2, A_\alpha^\dagger(\omega)A_\beta(\omega)\}\right],$$

$$(\mathrm{D}.1b)$$

which has the structure of $\mathrm{d}\rho^{(1)}(t)/\mathrm{d}t$ and so verifies the $2 \to 1$ descent.

To verify the $n \to n-1$ descent we make some simplifications in notation. We omit all superscripts (1), since this leads to no ambiguities, as well as all time arguments (t) and all frequency arguments (ω). We also abbreviate

$$L_\ell \equiv \sum_{\omega\alpha\beta} S_{\alpha\beta}(\omega)A_{\ell\alpha}^\dagger(\omega)A_{\ell\beta}(\omega),$$

$$M_\ell \equiv \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}(\omega)A_{\ell\alpha}^\dagger(\omega)A_{\ell\beta}(\omega). \tag{D.2a}$$

Our general strategy is to split the sum $\sum_{\ell=1}^{n}$ containing $(\rho_1 \cdots \rho_n)_\ell$ into $\sum_{\ell=2}^{n-1}$ plus the $\ell = 1$ and the $\ell = n$ terms, and to split the sum $\sum_{\ell=1}^{n}$ containing $(\rho_1 \cdots \rho_n)_{\ell\ell+1}$ into $\sum_{\ell=2}^{n-2}$ plus the $\ell = 1$, $\ell = n-1$ and $\ell = n$ terms. For the part of $\mathrm{d}\rho^{(n)}/\mathrm{d}t$ involving L_ℓ, we have

$$\sum_{\ell=2}^{n-1} (\rho_1 \cdots \rho_{n-1})_\ell \rho_n \mathrm{i}[\rho_\ell, L_\ell] + (\rho_2 \cdots \rho_n)\mathrm{i}[\rho_1, L_1] + (\rho_1 \cdots \rho_{n-1})\mathrm{i}[\rho_n, L_n], \tag{D.2b}$$

which on contracting the column index associated with the subscript n with the row index associated with the subscript 1, and relabelling all quantities that had subscript n with subscript 1, since they act now in the same Hilbert space, gives

$$\sum_{\ell=2}^{n-1} \left(\rho_1^2 \rho_2 \cdots \rho_{n-1}\right)_\ell \mathrm{i}[\rho_\ell, L_\ell] + \rho_2 \cdots \rho_{n-1}\mathrm{i}(\rho_1[\rho_1, L_1] + [\rho_1, L_1]\rho_1)$$

$$= \sum_{\ell=2}^{n-1} \left(\rho_1^2 \rho_2 \cdots \rho_{n-1}\right)_\ell \mathrm{i}[\rho_\ell, L_\ell] + \rho_2 \cdots \rho_{n-1}\mathrm{i}\left[\rho_1^2, L_1\right], \tag{D.2c}$$

which has the correct structure for the corresponding part of $\mathrm{d}\rho^{(n-1)}/\mathrm{d}t$, with ρ_1 replaced by ρ_1^2. The remainder of $\mathrm{d}\rho^{(n)}/\mathrm{d}t$ is

$$-\sum_{\ell=2}^{n-1} (\rho_1 \cdots \rho_n)_\ell \frac{1}{2}\{M_\ell, \rho_\ell\} - (\rho_2 \cdots \rho_n)\frac{1}{2}\{M_1, \rho_1\} - (\rho_1 \cdots \rho_{n-1})\frac{1}{2}\{M_n, \rho_n\}$$

$$+ \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}\left(\sum_{\ell=2}^{n-2} (\rho_1 \cdots \rho_n)_{\ell\ell+1}\rho_\ell A_{\ell\alpha}^\dagger A_{\ell+1\beta}\rho_{\ell+1} + \rho_3 \cdots \rho_n \rho_1 A_{1\alpha}^\dagger A_{2\beta}\rho_2\right.$$

$$\left. + \rho_1 \cdots \rho_{n-2}\rho_{n-1} A_{n-1\alpha}^\dagger A_{n\beta}\rho_n + \rho_2 \cdots \rho_{n-1}\rho_n A_{n\alpha}^\dagger A_{1\beta}\rho_1\right). \tag{D.3a}$$

Again, contracting the column index associated with the subscript n with the row index associated with the subscript 1, and relabelling all quantities that had subscript n with

subscript 1, since they act now in the same Hilbert space, gives

$$-\sum_{\ell=2}^{n-1}\left(\rho_1^2\cdots\rho_{n-1}\right)_\ell\frac{1}{2}\{M_\ell,\rho_\ell\}-(\rho_2\cdots\rho_{n-1})\left(\rho_1M_1\rho_1(*)+\frac{1}{2}\{M_1,\rho_1^2\}\right)$$

$$+\sum_{\omega\alpha\beta}\gamma_{\alpha\beta}\left(\sum_{\ell=2}^{n-2}\left(\rho_1^2\cdots\rho_{n-1}\right)_{\ell\ell+1}\rho_\ell A_{\ell\alpha}^\dagger A_{\ell+1\beta}\rho_{\ell+1}+(\rho_3\cdots\rho_{n-1})\rho_1^2A_{1\alpha}^\dagger A_{2\beta}\rho_2\right.$$

$$\left.+\rho_2\cdots\rho_{n-2}\rho_{n-1}A_{n-1\alpha}^\dagger A_{1\beta}\rho_1^2+\rho_2\cdots\rho_{n-1}\rho_1A_{1\alpha}^\dagger A_{1\beta}\rho_1(*)\right),\tag{D.3b}$$

which on cancelling the terms marked with $(*)$ gives the corresponding part of $d\rho^{(n-1)}/dt$, with ρ_1 replaced by ρ_1^2. This completes the verification of the $n\to n-1$ descent.

Appendix E. Descent equations for the Caldeira–Leggett model

We verify here that equations (56a) and (56b) obey the descent equations of equation (34). As in the preceding appendix, we simplify the notation by omitting all superscripts (1) and all time arguments (t). We first verify the $n=2$ to $n=1$ descent. For the $n=2$ case of equation (56b), we have

$$D\rho^{(2)}/dt=\rho_2\left[-2m\gamma k_BT\left(x_1^2\rho_1+\rho_1x_1^2\right)+i\gamma\left(\rho_1p_1x_1-x_1p_1\rho_1\right)\right]$$

$$+\rho_1\left[-2m\gamma k_BT\left(x_2^2\rho_2+\rho_2x_2^2\right)+i\gamma\left(\rho_2p_2x_2-x_2p_2\rho_2\right)\right]$$

$$+4m\gamma k_BT\left(\rho_1x_1x_2\rho_2+\rho_2x_2x_1\rho_1\right)+i\gamma\left[\rho_1(x_1p_2-p_1x_2)\rho_2\right.$$

$$\left.+\rho_2(x_2p_1-p_2x_1)\rho_1\right].\tag{E.1a}$$

Contracting the column index associated with the subscript 1 with the row index associated with the subscript 2, and dropping subscripts since all operators now act in the same Hilbert space, we get

$$D\rho^{(2)}/dt\to-2m\gamma k_BT(x^2\rho^2+\rho x^2\rho)+i\gamma(\rho px\rho-xp\rho^2)$$

$$-2m\gamma k_BT(\rho x^2\rho+\rho^2x^2)+i\gamma(\rho^2px-\rho xp\rho)$$

$$+4m\gamma k_BT(\rho x^2\rho+x\rho^2x)+i\gamma[\rho(xp-px)\rho+p\rho^2x-x\rho^2p].\tag{E.1b}$$

We see that the terms that have an operator sandwiched between two factors of ρ cancel, leaving only terms involving ρ^2, which have the form of equation (56a) with ρ replaced by ρ^2.

To check the $n>2$ to $n-1$ descent, we split the sums that occur in the same manner as in appendix D. We thus write equation (56b) in the form

$$D\rho^{(n)}/dt=\sum_{\ell=2}^{n-1}(\rho_1\cdots\rho_n)_\ell\left[-2m\gamma k_BT\left\{x_\ell^2,\rho_\ell\right\}+i\gamma\left(\rho_\ell p_\ell x_\ell-x_\ell p_\ell\rho_\ell\right)\right]$$

$$+\rho_2\cdots\rho_n\left[-2m\gamma k_BT\left\{x_1^2,\rho_1\right\}+i\gamma\left(\rho_1p_1x_1-x_1p_1\rho_1\right)\right]$$

$$+\rho_1\cdots\rho_{n-1}\left[-2m\gamma k_BT\left\{x_n^2,\rho_n\right\}+i\gamma\left(\rho_np_nx_n-x_np_n\rho_n\right)\right]$$

$$+\sum_{\ell=2}^{n-2}(\rho_1\cdots\rho_n)_{\ell\ell+1}[4m\gamma k_BT\rho_\ell x_\ell x_{\ell+1}\rho_{\ell+1}+i\gamma\rho_\ell(x_\ell p_{\ell+1}-p_\ell x_{\ell+1})\rho_{\ell+1}]$$

$$+\rho_3\cdots\rho_n[4m\gamma k_BT\rho_1x_1x_2\rho_2+i\gamma\rho_1(x_1p_2-p_1x_2)\rho_2]$$

$$+\rho_1\cdots\rho_{n-2}[4m\gamma k_BT\rho_{n-1}x_{n-1}x_n\rho_n+i\gamma\rho_{n-1}(x_{n-1}p_n-p_{n-1}x_n)\rho_n]$$

$$+\rho_2\cdots\rho_{n-1}[4m\gamma k_BT\rho_nx_nx_1\rho_1+i\gamma\rho_n(x_np_1-p_nx_1)\rho_1].\tag{E.2}$$

We now contract the column index associated with the subscript n with the row index associated with the subscript 1, and relabel all quantities that had subscript n with subscript 1, since they act now in the same Hilbert space. As is readily seen by inspection of equation (E.2), this gives equation (56b) with n replaced by $n-1$ and with ρ_1 replaced by ρ_1^2, together with terms of the wrong structure, that grouped together give $(4-2-2)\rho_2 \cdots \rho_{n-1} m\gamma k_B T \rho_1 x_1^2 \rho_1 = 0$ and $(1-1)\rho_2 \cdots \rho_{n-1} \mathrm{i}\gamma \rho_1 (x_1 p_1 - p_1 x_1)\rho_1 = 0$, which thus vanish. This completes the verification of the descent equation for equation (56b).

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